More on Covariance

Gene Quinn

Variance-Covariance Matrix

Definition: The variance-covariance matrix for two jointly distributed random variables X_1, X_2 is the matrix V with:

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- The off diagonal elements are $\sigma_{ij}, \; i \neq j \; = \;$ Cov(X_i, X_j)

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With four variates,

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \sigma_{34} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_4^2 \end{bmatrix}$$

Definition

If the variates are independent, all covariances are zero and V is diagonal:

$$V = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

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- covariance Cov$(X_1, X_2) = \sigma_{12}$
- variances Var$(X_1) = \sigma_1^2$ andVar$(X_2) = \sigma_2^2$

the **correlation coefficient** of X_1 and X_2, denoted by $\rho(X_1, X_2)$, is given by

$$\rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

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$$\rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

Note: $\sigma_1 = \sqrt{\sigma_1^2}$ and $\sigma_2 = \sqrt{\sigma_2^2}$

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A negative correlation means that high values of X_1 tend to be associated with *low* values of X_2.

The Correlation Matrix

Definition: The **correlation matrix** for jointly distributed random variables $X_1, X_2, \ldots, X_n$ is the matrix R with:

- i^{th} diagonal element 1
- off diagonal elements $\rho_{ij},\ i \neq j\ =\ \frac{\text{Cov}(X_i, X_j)}{\sigma_i \sigma_j}$

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For a bivariate distribution,

$$R \ = \ \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix}$$

The Variance and Correlation Matrices

The variance-covariance matrix and the correlation matrix are related in the following way:

Suppose $\vec{X} = (X_1, x_2, \ldots, X_n)$ is a vector of random variables with variance-covariance matrix V. Let S be the diagonal matrix with i^{th} element equal to the *reciprocal* of $\sqrt{\sigma_i^2}$

$$S = \begin{bmatrix} \frac{1}{\sqrt{\sigma_1^2}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\sigma_2^2}} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\sqrt{\sigma_1^2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{\sigma_n^2}} \end{bmatrix}$$

The Variance and Correlation Matrices

Then the variance matrix V and correlation matrix R satisfy:

$$SVS = R$$

Linear Combinations

Suppose $\vec{X}$ is a vector of random variables and $\vec{a}$ is a vector of weights.

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The variance of the linear combination

$$\vec{a} \cdot \vec{X}$$

is the quadratic form:

$$\vec{a}'V\vec{a}$$

where a' is represents the (column) vector a written as a row vector.

Linear Combinations

Suppose $\vec{X} = \{X_1, X_2, X_3\}$ is a vector of random variables with variance matrix

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}$$

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Find the variance of the linear combination $2X_1 - X_2 + X_3$

In this case, $\vec{a} = \{2\ -1\ 1\}$, and the variance of the linear combination is the quadratic form:

$$\vec{a}'V\vec{a} = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Linear Combinations

First carry out the multiplication of $\vec{a}'V$

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$$\begin{bmatrix} (2\sigma_1^2 - \sigma_{12} + \sigma 13) & (2\sigma_{12} - \sigma_2^2 + \sigma_{23}) & (2\sigma_{13} - \sigma_{23} + \sigma_3^2) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

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$$\left[\begin{array}{ccc} (2\sigma_1^2 - \sigma_{12} + \sigma 13) & (2\sigma_{12} - \sigma_2^2 + \sigma_{23}) & (2\sigma_{13} - \sigma_{23} + \sigma_3^2) \end{array}\right] \left[\begin{array}{c} 2 \\ -1 \\ 1 \end{array}\right.$$

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$$\vec{a}'V\vec{a} \;=\; (2)(2\sigma_1^2 - \sigma_{12} + \sigma_{13}) \;+\; (-1)(2\sigma_{12} - \sigma_2^2 + \sigma_{23})$$
$$+(1)(2\sigma_{13} - \sigma_{23} + \sigma_3^2)$$

Linear Combinations

$$\begin{bmatrix} (2\sigma_1^2 - \sigma_{12} + \sigma 13) & (2\sigma_{12} - \sigma_2^2 + \sigma_{23}) & (2\sigma_{13} - \sigma_{23} + \sigma_3^2) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{a}'V\vec{a} = (2)(2\sigma_1^2 - \sigma_{12} + \sigma_{13}) + (-1)(2\sigma_{12} - \sigma_2^2 + \sigma_{23}) + (1)(2\sigma_{13} - \sigma_{23} + \sigma_3^2)$$

$$\vec{a}'V\vec{a} = 4\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 4\sigma_{12} + 4\sigma_{13} - 2\sigma_{23}$$