Name:

1) A random variable Y has density function

$$f(y) = \begin{cases} a \cdot (2-y) & \text{if} \quad -2 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the value of a that makes f a valid density function. The density function has to integrate to one over its interval of support:

$$\int_{-2}^{2} a \cdot (2-y)dy = 8a = 1$$

so $a = 1/8$.

b) Find the cumulative distribution function $F(y)$ By definition the CDF is

$$F(y) = \int_{-2}^{y} \frac{2-t}{8} dt = \frac{3}{4} + \frac{y}{4} - \frac{y^2}{16}$$

c) Find the expected value $E(Y)$

$$E(Y) = \int_{-2}^{2} \frac{2-y}{8} dy = -\frac{2}{3}$$

d) Find the variance $V(Y)$

$$E(Y^2) = \int_{-2}^{2} \frac{2-y}{8} dy = \frac{4}{3} \quad V(Y) = E(Y^2) - [E(Y)]^2 = \frac{4}{3} - \left(-\frac{2}{3}\right)^2 = \frac{8}{9}$$

2) A random variable Y has density function

$$f(y) = \begin{cases} k \cdot e^{-y} & \text{if} \quad y \in [1, \infty) \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the value of k that makes f a valid density function. The density must integrate to one over its support:

$$\int_{1}^{\infty} k \cdot e^{-y} dy \quad = \quad -ke^{-y}\Big|_1^{\infty} \; = \; ke^{-1} = 1$$

so $k = e$.

b) Find the cumulative distribution function $F(y)$

$$F(y) = \int_1^y f(t)dt = 1 - e^{1-y}$$

c) Find the expected value $E(Y)$

$$E(Y) = \int_1^\infty y \cdot ee^{-y}dy = 2$$

d) Find the variance $V(Y)$

$$E(Y^2) = \int_1^\infty y^2 \cdot ee^{-y}dy = 5 \quad V(Y) = E(Y^2) - [E(Y)]^2 = 1$$

e) Find the moment-generating function $m(t)$ By definition,

$$m_y(t) = \int_1^\infty e^{ty}e \cdot e^{-y}dy = \frac{e^t}{1-t}$$

3) A random variable Y has density function

$$f(y) = \begin{cases} k & \text{if} \quad y \in [1,5] \\ 0 & \quad \text{elsewhere} \end{cases}$$

a) Find the value of k that makes f a valid density function. The density must integrate to one over its support:

$$\int_1^5 k\,dy = ky|_1^5 = 4k = 1$$

so $k = 1/4$.

b) Find the cumulative distribution function $F(y)$

$$F(y) = \int_1^y \frac{1}{4}dt = \frac{y}{4} - \frac{1}{4}$$

(originally posted incorrectly)

c) Find the expected value $E(Y)$

$$E(Y) = \int_1^5 y \cdot \frac{1}{4} = \left.\frac{y^2}{8}\right|_1^5 = \frac{25}{8} - \frac{1}{8} = 3$$

d) Find the variance $V(Y)$

$$E(Y^2) = \int_1^5 \frac{y^2}{4}dy = \left.\frac{y^3}{12}\right|_1^5 = \frac{31}{3}$$

$$V(Y) = E(Y^2) - [E(Y)]^2 \quad = \quad \frac{31}{3} - 3^2 = \frac{4}{3}$$

4) A random variable Y has cumulative distribution function

$$F(y) = \begin{cases} y^{3/2} & \text{if} \quad y \in [0,1] \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the density function $f(y)$. The density function is the derivative of the cumulative distribution function:

$$f(y) = \frac{d}{dy} y^{3/2} = \frac{3}{2}\sqrt{y}$$

b) Find the expected value $E(Y)$

$$E(Y) = \int_0^1 y \cdot \frac{3}{2}\sqrt{y} = \frac{3}{5}$$

c) Find the variance $V(Y)$

$$E(Y^2) = \int_0^1 y^2 \cdot \frac{3}{2}\sqrt{y} = \frac{3}{7}$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{3}{7} - \left(\frac{3}{5}\right)^2 = \frac{12}{175}$$

d) Find the median $\phi_{.5}$ The median satisfies the equation

$$F(y) = y^{3/2} = 0.5$$

so

$$y^3 = 0.5^2 \quad \text{and} \quad y = \sqrt[3]{.25}$$

5) Suppose Y has expected value $E(Y) = \mu = 50$ and variance $V(Y) = 16$.

a) Find an upper bound for the probability that Y takes a value outside the interval $[38, 62]$ This is an interval of half-width $12 = 3\sqrt{16} = 3\sigma$, so $k = 3$ and the probability is at most

$$\frac{1}{k^2} = \frac{1}{9}$$

b) Find a lower bound for the probability that Y takes a value in the interval $[34, 66]$ This is an interval of half-width 4σ so $k = 4$ and the probability that Y falls in this interval is at least

$$1 - \frac{1}{k^2} = 1 - \frac{1}{16} = \frac{15}{16}$$

c) Find a value d such that the probability that Y takes a value outside the interval $[50 - d, 50 + d]$ is less than or equal to $1/10$. In this case we need to find k such that

$$\frac{1}{k^2} = \frac{1}{10} \quad \text{so} \quad k = \sqrt{10} \quad \text{and} \quad d = \sigma\sqrt{10} = 4\sqrt{10}$$