Averages

Gene Quinn

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Given a finite set of numbers

$$S = \{x_1, x_2, \ldots, x_n\}$$

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Example: Suppose

$$S = \{1, 2, 3, 4, 5, 6, 7\}$$

then:

$$\overline{x} = \frac{\sum_i x_i}{n} = \frac{1+2+3+4+5+6+7}{7} = \frac{28}{7} = 4$$

Weighted Averages

A generalization of the average is the **weighted average**.

Suppose we have a finite set of numbers x_i as before,

$$S = \{x_1, x_2, \ldots, x_n\}$$

In addition, we have a set of weights w_i,

$$W = \{w_1, w_2, \ldots, w_n\}$$

we define the **weighted average** of the numbers in S as

$$\overline{x}_w = \frac{\sum_i w_i \cdot x_i}{\sum_i w_i}$$

Weighted Averages

Example: Suppose

$$S = \{1, 2, 3, 4, 5, 6, 7\} \quad \text{and} \quad W = \{1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1\}$$

Weighted Averages

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$$S = \{1,2,3,4,5,6,7\} \quad \text{and} \quad W = \{1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1\}$$

then:

$$\overline{x}_w = \frac{\sum_i w_i \cdot x_i}{\sum_i w_i} = \frac{1 \cdot 1 + \frac{1}{2} \cdot 2 + 1 \cdot 3 + \frac{1}{2} \cdot 4 + 1 \cdot 5 + \frac{1}{2} \cdot 6 + 1 \cdot 7}{1 + \frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{1}{2} + 1}$$

or

$$\overline{x}_w = \frac{22}{\left(\frac{11}{2}\right)} = \frac{22 \cdot 2}{11} = 4$$

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or

$$\overline{x}_w = \frac{22}{\left(\frac{11}{2}\right)} = \frac{22 \cdot 2}{11} = 4$$

(The fact that $\overline{x}_w$ is the same as $\overline{x}$ from the previous example is coincidence. Usually a weighted average will differ from the unweighted average.)

Weighted Averages with Normalized Weights

Suppose we have a weighted average as before,

$$\overline{x}_w = \frac{\sum_i w_i \cdot x_i}{\sum_i w_i}$$

Replace each w_i with $v_i = a \cdot w_i$, where a is an arbitrary nonzero constant.

Now the weighted average becomes

$$\overline{x}_v = \frac{\sum_i v_i \cdot x_i}{\sum_i v_i} = \frac{\sum_i a \cdot w_i \cdot x_i}{\sum_i a \cdot w_i} = \frac{a \sum_i w_i \cdot x_i}{a \sum_i w_i} = \frac{\sum_i w_i \cdot x_i}{\sum_i w_i} = \overline{x}_w$$

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This result shows that we can scale the weights any way we choose without changing the result.

Weighted Averages with Normalized Weights

The most beneficial choice would be to scale the weights so that

$$\sum_i w_i \; = \; 1$$

and in this case the formula

$$\overline{x}_w \; = \; \frac{\sum_i w_i \cdot x_i}{\sum_i w_i}$$

simplifies to

$$\overline{x}_w \; = \; \sum_i w_i \cdot x_i$$

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It's always possible to choose weights that sum to 1; Let

$$a \;=\; \frac{1}{\sum_i w_i}$$

The Continuous Case

Define the average of a continuous function f on an interval $[a, b]$ as

$$\overline{x} = \frac{\int_a^b f(x)\,dx}{b-a}$$

This is a reasonable definition because $\overline{x}$ is the height of a rectangle with base $b-a$ whose area is the same as the area under the graph of f from $x = a$ to $x = b$.

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Example: The average value of $f(x) = x^2$ over the interval from 0 to 2 will be:

$$\overline{x} = \frac{\int_0^2 x^2\,dx}{2-0} = \frac{\left.\frac{x^3}{3}\right|_0^2}{2} = \frac{\frac{8}{3} - \frac{0}{3}}{2} = \frac{4}{3}$$

Weighted Averages of Continuous Functions

Suppose f is continuous on the interval $[a, b]$ and a weight function w is also continuous on $[a, b]$.

Define the weighted average of f on the interval as

$$\overline{x}_w = \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx}$$

Weighted Averages of Continuous Functions

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Define the weighted average of f on the interval as

$$\overline{x}_w = \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx}$$

Example: The average value of $f(x) = x^2$ with weight function $w(x) = x$ over the interval from 0 to 2 will be:

$$\overline{x}_w = \frac{\int_0^2 x \cdot x^2\, dx}{\int_0^2 x\, dx} = \frac{\left.\frac{x^3}{3}\right|_0^2}{\left.\frac{x^2}{2}\right|_0^2} = \frac{\frac{8}{3} - \frac{0}{3}}{\frac{4}{2} - \frac{0}{2}} = \frac{4}{3}$$

Normalized Weights

Suppose we have a weighted average of a continuous function f on an interval $[a, b]$ with (continuous) weight function $w(x)$. Then

$$\overline{x}_w \;=\; \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx}$$

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$$\overline{x}_w = \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx}$$

Consider what happens if we use the weight function $v(x) = c \cdot w(x)$ where c is some nonzero constant:

$$\overline{x}_v = \frac{\int_a^b v(x) \cdot f(x)\, dx}{\int_a^b v(x)\, dx} = \frac{\int_a^b c \cdot w(x) \cdot f(x)\, dx}{\int_a^b c \cdot w(x)\, dx}$$

$$= \frac{c \int_a^b w(x) \cdot f(x)\, dx}{c \int_a^b w(x)\, dx} = \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx} = \overline{x}_w$$

Normalized Weights

As with the discrete case, we can simplify the formula for the weighted average

$$\overline{x}_w = \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx}$$

by choosing a weight function with

$$\int_a^b w(x)\, dx = 1$$

Normalized Weights

As with the discrete case, we can simplify the formula for the weighted average

$$\overline{x}_w = \frac{\int_a^b w(x) \cdot f(x)\, dx}{\int_a^b w(x)\, dx}$$

by choosing a weight function with

$$\int_a^b w(x)\, dx = 1$$

Now the weighted average formula simplifies to:

$$\overline{x}_w = \int_a^b w(x) \cdot f(x)\, dx$$