Name:

1) Suppose $Y_1, \ldots, Y_6$ is a random sample of size 6 from a Weibull distribution

$$f(y) = \frac{m}{\alpha} y^{m-1} e^{-y^m/\alpha}, \quad y > 0$$

Show that if m is known, then

$$\sum_{i=1}^{6} y_i^m \quad \text{is sufficient for } \alpha$$

The likelihood function of the sample is:

$$L(\alpha) = \prod_{i=1}^{6} \frac{m}{\alpha} y_i^{m-1} e^{-y_i^m/\alpha} = \left[m^6 \left(\prod_{i=1}^{6} y_i \right)^{m-1} \right] \left[\frac{1}{\alpha^6} \exp\left(-\frac{\sum_{i=1}^{6} y_i^m}{\alpha} \right) \right]$$

Identify the first factor as $h(y_1, \ldots, y_6)$ and the second as $g(u, \alpha)$ with $u = \sum y_i^m$. By the Neyman factorization theorem, u is sufficient for α.

2) Let $Y_1, \ldots, Y_n$ be a random sample of size n from a geometric distribution. Use the factorization theorem to show that $\overline{Y}$ is sufficient for p.

The likelihood function of the sample is

$$L(p) = \prod_{i=1}^{n} p(1-p)^{y_i - 1} = p^n (1-p)^{\sum y_i - n} = p^n (1-p)^{n(\overline{y}-1)}$$

Identify $L(p)$ as $g(\overline{y}, p)$ and let $h(y_1, \ldots, y_n) = 1$. Then by the factorization theorem, $\overline{y}$ is sufficient for p.

3) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample and each Y_i has density function

$$f(y) = \left(\frac{2y}{\theta} \right) e^{-y^2/\theta} \quad y > 0$$

a) Show that $\sum_{i=1}^{n} y_i^2$ is sufficient for θ.

(See Example 9.7 in the text) The likelihood function of the sample is

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{2y_i}{\theta} \right) e^{-y_i^2/\theta} = \left[2^n \prod_{i=1}^{n} y_i \right] \left[\frac{1}{\theta^n} \exp\left(-\frac{\sum_{i=1}^{n} y_i^2}{\theta} \right) \right]$$

Identify the second factor as $g(u, \theta)$ and the first as $h(y_1, \ldots, y_n)$. Then by the factorization theorem, $u = \sum y_i^2$ is sufficient for θ.

b) Show that

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} y_i^2$$

is a MVUE of θ (hint: show that $\hat{\theta}$ is a function of a sufficient statistic and is unbiased)

Let $W = Y_i^2$. Then by the transform method, the density function of W is

$$f_W(w) = f(\sqrt(w))\frac{d\sqrt(w)}{dw} = \left(\frac{2}{\theta}\right)\left(\sqrt{w}e^{-w/\theta}\right)\left(\frac{1}{2\sqrt{w}}\right) = \left(\frac{1}{\theta}\right)e^{-w/\theta}, \quad w > 0$$

so $W = Y_i^2$ has an exponential distribution with parameter θ. This means that $E(\overline{W}) = \theta$ and since

$$\overline{W} = \frac{1}{n}\sum_{i=1}^{n} w_i = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

so

$$\hat{\theta}\frac{1}{n}\sum_{i=1}^{n} Y_i^2$$

is an unbiased estimator of θ that is a function of the sufficient statistic, and is therefore an MVUE of θ.

4) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample and each Y_i has density function

$$f(y) = \alpha\beta^{\alpha}y^{-(\alpha+1)} \quad y > 0$$

Show that if β is known then $\prod_{i=1}^{n} y_i$ is sufficient for α.

The likelihood function of the sample is:

$$L(\alpha) = \prod_{i=1}^{n} \alpha\beta^{\alpha}y_i^{-(\alpha+1)} = \alpha^n\beta^{n\alpha}\left(\prod_{i=1}^{n} y_i\right)^{-(\alpha+1)}$$

Identify $L(\alpha)$ as $g(u, \alpha)$ with $u = \prod y_i$, and $h(y_1, \ldots, y_n) = 1$. Then by the Neyman factorization theorem, $\prod y_i$ is sufficient for α.