Name:

1) Suppose $(Y_1, \ldots, Y_5)$ is a random sample of size 5 from a $N(\mu, 1)$ distribution, and that

$$\hat{\mu}_1 = Y_1 + Y_3 + Y_5 - Y_2 - Y_4 \quad \text{and} \quad \hat{\mu}_2 = \overline{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$$

Show that $\hat{\mu}_1$ and $\hat{\mu}_2$ are unbiased and find the efficiency of $\hat{\mu}_1$ relative to $\hat{\mu}_2$.

Note that both estimators are linear combinations of the sample values, with

$$t_1' = (1, -1, 1, -1, 1) \quad \text{and} \quad t_2' = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$$

Since

$$\mu' = (\mu, \mu, \mu, \mu, \mu) \quad \text{and} \quad V = I$$

we get

$$E(\hat{\mu}_1) = t_1'\mu = 3\mu - 2\mu = \mu \quad \text{and} \quad E(\hat{\mu}_2) = t_2'\mu = 5\frac{\mu}{5} = \mu$$

so both are unbiased and

$$V(\hat{\mu}_1) = t_1'Vt_1 = t_1't_1 = 5 \quad \text{and} \quad V(\hat{\mu}_2) = t_2't_2 = 5\frac{1}{5^2} = \frac{1}{5}$$

and the relative efficiency is

$$\text{eff}(\hat{\mu}_1, \hat{\mu}_2) = \frac{\hat{\mu}_1}{\hat{\mu}_2} = \frac{1}{25}$$

2) If Y has a a binomial distribution with n trials and probability of success p, show that Y/n is a consistent estimator for p.

From Table 8.1, recall that

$$V(\hat{p}) = V(Y/n) = \frac{pq}{n} \to 0 \quad \text{as} \quad n \to \infty$$

So by Theorem 9.1 this estimator is consistent.

3) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample and each Y_i has density function

$$f(y) = \begin{cases} \theta y^{\theta-1} & \text{if} \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Show that $\overline{Y}$ is a consistent estimator for $\theta/(\theta+1)$.

Comparing the density function of Y to a beta distribution, we see that Y has a beta density with $\alpha = \theta$ and $\beta = 1$.

From the results in the back cover of the text,

$$E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{\theta}{\theta+1} \quad \text{and} \quad V(Y) = \frac{\theta}{(\theta+2)(\theta+1)^2)}$$

Using the results for expected values and variances of linear combinations, which are true regardless of the distributon of Y, we can say that:

$$E(\overline{Y}) = E(Y) = \frac{\theta}{\theta+1} \quad \text{and} \quad V(\overline{Y}) = \frac{\theta}{n(\theta+2)(\theta+1)^2}$$

Since $\overline{Y}$ is unbiased and its variance approaches zero as the sample size becomes large, $\overline{Y}$ is a consistent estimator for $E(Y) = \theta/(\theta+1)$.

4) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a gamma distribution with parameters α and β. Show that $\overline{Y}$ converges in probability to a constant and find that constant.

From our results for linear combinations of random variables, we know that

$$E(\overline{Y}) = E(Y) = \alpha\beta$$

and

$$V(\overline{Y}) = \frac{1}{n} V(Y)$$

Since $\overline{Y}$ is an unbiased estimate and its variance tends to zero, $\overline{Y}$ is a consistent estimator for $\alpha\beta$, so $\overline{Y}$ converges in probability to $\alpha\beta$.

5) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample from the probability density function

$$f(y) = \begin{cases} \frac{2}{y^2} & \text{if} \quad y \ge 2 \\ 0 & otherwise \end{cases}$$

Determine whether or not the law of large numbers applies to $\overline{Y}$.

Note that

$$E(Y^2) = \int_2^\infty y^2 \cdot \frac{2}{y^2} dy$$

Since the integral diverges, $E(Y^2)$ is infinite so the variance of Y does not exist and the law of large numbers does not apply.