The Bernoulli Distribution

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Bernoulli Trials

Definition: A **Bernoulli trial** is an experiment which:

1) Has exactly two outcomes, usually called *success* and *failure*

and

2) Has a fixed probabilities p associated with the outcome "success" and $q = 1 - p$ associated with the outcome "failure".

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2) Has a fixed probabilities p associated with the outcome "success" and $q = 1 - p$ associated with the outcome "failure".

The sample space for a Bernoulli trial experiment contains two elements:

$$S = \{\text{success}, \text{failure}\}$$

Bernoulli Trials

Recall that an *event* is any subset of a sample space.

In general, a finite sample space with n elements has 2^n possible subsets (Including S, each possible outcome, and the empty set $\emptyset$), so in the case of a Bernoulli trial there are four possible events.

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The following table lists the four possible events and defines a probability function by associating a real number with each event:

Event	Description	Probability
success $\cup$ failure	either success or failure occurs	1
success	the outcome is success	p
failure	the outcome is failure	$q = 1 - p$
$\emptyset$	neither success nor failure occurs	0

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The fourth axiom is not required in this case because the sample space is finite.

Bernoulli Trials

Usually a random variable X is associated with a Bernoulli trial by the following definition.

$$X = \begin{cases} 0 & \text{if the outcome is "failure"} \\ 1 & \text{if the outcome is "success"} \end{cases}$$

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$$X = \begin{cases} 0 & \text{if the outcome is "failure"} \\ 1 & \text{if the outcome is "success"} \end{cases}$$

Recall that a *random variable* is a real-valued function whose domain is a sample space, so the above definition qualifies since it is a real-valued function defined on the sample space of a Bernoulli trial,

$$S = \{\text{success}, \text{failure}\}$$

Bernoulli Trials

Finally, we will define a **probability density function** or pdf for the random variable X defined by:

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A *discrete probability density function* $p_X(k)$ maps the values of a discrete random variable X into $[0, 1]$ according to the rule

$$p_X(k) = P\left(\{s \in S \,|\, X(s) = k\}\right)$$

Bernoulli Trials

It is common to simplify the expression

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The resulting simplified definition for the probability density function is:

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In the case of a Bernoulli trial, the pdf is usually defined as

$$p_X(k) \;=\; \begin{cases} p & \text{if} \quad k = 1 \\ q = 1 - p & \text{if} \quad k = 0 \end{cases}$$

Related Probability Distributions

A number of important probability distributions arise when sequences of independent Bernoulli trials with a constant probability of success p are performed.

The most important is the result of performing exactly n Bernoulli trials, each independent of the others and having the same probability of success p.

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If X_i is the random variable associated with the i^{th} Bernoulli trial, for $1 \leq i \leq n$, then we define a **binomial random variable** Y to be

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$$Y = \sum_{i=1}^{n} X_i$$

Recalling that $X_i = 1$ if the i^{th} Bernoulli trial results in success, we interpret Y as the number of successes in n independent Bernoulli trials with probability of success equal to p.

Related Probability Distributions

We will see that the probability density function of the binomial random variable Y obtained by adding the results X_i of n independent Bernoulli trials with probability of success p is:

$$p_Y(k) \;=\; \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

Related Probability Distributions

Another related distribution arises from the following experiment:

Independent Bernoulli trials each having probability of success p are performed until the first success occurs. number of important probability distributions arise when sequences of independent

A random variable Y is defined with the following pdf:

$$p_Y(k) \;=\; P\left(\text{first success occurs on trial } k\right), \quad k = 1, 2, 3, \ldots$$

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The distribution of Y in this case is called the **geometric distribution**

We will see that the probability density function for the geometric distribution is:

$$p_Y(k) \;=\; (1-p)^{k-1} \cdot p, \quad k = 1, 2, \ldots$$

Related Probability Distributions

Suppose instead we conduct independent Bernoulli trials with probability of success p until we obtain r successes, for some integer $r > 1$.

Define a random variable Y with the following pdf:

$$p_Y(k) \ = \ P\left(\text{the } r^{th} \text{ success occurs on trial } k\right), \quad k = r, r+1, \ldots$$

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$$p_Y(k) \;=\; P\left(\text{the } r^{th} \text{ success occurs on trial } k\right), \quad k = r, r+1, \ldots$$

The distribution of Y in this case is called the **negative binomial distribution**

We will see that the probability density function for the negative binomial distribution is:

$$p_Y(k) \;=\; \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$