MA396 Takehome Exam 1

Name:

(Note: if you use a computer algebra system, please include the output with your exam)

1) If two continuous random variables X and Y have joint density function $f_{XY}(x, y)$, define the **covariance** of X and Y as

$$\sigma_{xy} = \text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y)$$

Show that if X and Y are independent random variables with expectations $\text{E}(X) = \overline{x}$ and $\text{E}(Y) = \overline{y}$, then $\sigma_{xy} = 0$.

2) A distribution is said to be **memoryless** if

$$P(Y > s + t \mid Y > t) = P(Y > s)$$

Show that a random variable Y with the exponential distribution

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \in [0, \infty)$$

has the memoryless property.

3) Suppose Y is a random variable with $\mathrm{E}(Y) = \mu$ and $\mathrm{Var}(Y) = \sigma^2$, and that $\mathrm{E}(Y^3)$ exists.

a) Show that

$$\mathrm{E}(Y-\mu)^2 \;=\; \mathrm{E}(Y^2) - \mu^2$$

b) Show that

$$\mathrm{E}(Y-\mu)^3 \;=\; \mathrm{E}(Y^3) - 3\mu\sigma^2 - \mu^3$$

4) Suppose $\vec{X} = (X_1, X_2, \ldots, X_n)$ is a vector or n random variables with joint density

$$f_X(x_1, x_2, \ldots, x_n), \quad x_i \in \mathbb{R}$$

Let $\vec{X}_k = (X_1, X_2, \ldots, X_k)$ represents the first k components of $\vec{X}$. Then the *(joint) marginal density* of $\vec{X}_k$ is:

$$g_{X_k}(x_1, x_2, \ldots, x_k) \;=\; \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_X(x_1, x_2, \ldots, x_n)\, dx_{k+1} \cdots dx_n$$

In general, given a jointly distributed vector of n random variables $\vec{X}$, there will be a joint marginal density for every nonempty proper subset of $\{X_1, X_2, \ldots, X_n\}$. This density function will be defined by an integral similar to the one above.

If $\vec{X} = (X_1, X_2, X_3)$ has joint density function

$$f_X(x_1, x_2, x_3) \;=\; \frac{1}{9}(x_1 + x_1^2 x_2)e^{-x_3}, \quad x_1 \in [0,3]\,, x_2 \in [0,1],\; x_3 \in [0,\infty)$$

find the six possible marginal density functions.

5) Suppose $\vec{X} = (X_1, X_2, \ldots, X_n)$ is a vector or n random variables with joint density

$$f_X(x_1, x_2, \ldots, x_n), \quad x_i \in \mathbb{R}$$

Let $\vec{X}_k = (X_1, X_2, \ldots, X_k)$ represent the first k components of $\vec{X}$ and $\vec{X}_{n-k}$ represent the last $n-k$ components of $\vec{X}$. Then the *conditional density* of $\vec{X}_k$ given $\vec{X}_{n-k}$ is:

$$f(x_1, x_2, \ldots, x_k \,|\, x_{k+1}, \ldots, x_n) \;=\; \frac{f_X(x_1, x_2, \ldots, x_n)}{g_{X_{n-k}}(x_{k+1}, x_{k+2}, \ldots, x_n)}$$

If $\vec{X} = (X_1, X_2, X_3)$ has joint density function given in problem 4), find the six possible conditional density functions.

6) For the joint density function in problem 4), find the variance-covariance matrix of $\vec{X}$.

7) For the joint density function in problem 4), find the variance-covariance matrix of the random vector $\vec{Y} = \{Y_1, Y_2\}$ defined by:

$$\begin{aligned} Y_1 &= X_1 + X_2 - 2X_3 \\ Y_2 &= X_1 - X_2 + X_3 \end{aligned}$$

8) Suppose a random variable Y has density function

$$f_Y(y) = \frac{k}{1+y^2}, \quad y \in \mathbb{R}$$

a) Find the value of k.

b) Show that $\mathrm{E}(Y)$ does not exist.

9) Suppose $\vec{X} = (X_1, X_2, X_3)$ is a vector of random variables with joint density function

$$f(x_1, x_2, x_3) \;=\; \exp(-x_1 - x_2 - x_3), \quad x_1, x_2, x_3 \in [0, \infty)$$

Define the *multivariate moment-generating function* $M_X(t_1, t_2, t_3)$ as

$$M_X(t_1, t_2, t_3) \;=\; \mathrm{E}(\exp(t_1 x_1 + t_2 x_2 + t_3 x_3)$$

Show that

$$M_X(t_1, t_2, t_3) \;=\; -\frac{1}{(t_1 - 1)(t_2 - 1)(t_3 - 1)}$$

10) The density function for the multivariate normal distribution is:

$$f(x_1, x_2, \ldots, x_n) \;=\; \frac{1}{\sqrt{(2\pi)^n |V|}} \exp\left(-\frac{(x-\mu)' V^{-1} (x-\mu)}{2}\right)$$

Where:

$$\mu \;=\; (\mu_1, \mu_2, \ldots, \mu_n)$$

is a vector of means, V is an $n \times n$ variance-covariance matrix, and $|V|$ is the matrix determinant of V.

When $n = 3$,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \text{ is a vector of random variables}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \text{ is a vector of expected values or means}$$

$$x - \mu = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ x_3 - \mu_3 \end{bmatrix} \text{ is a vector of deviations from means}$$

and the variance-covariance matrix is

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}$$

The moment-generating function of the multivariate normal distribution is

$$M_X(t) = \exp(t'x + t'Vt)$$

where $t = (t_1, t_2, \ldots, t_n)$ is a vector of parameters. When $n = 3$,

$$t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

The moment-generating function of the (joint) marginal distribution of X_1 and X_2 is obtained by setting $t_3 = 0$ in the multivariate normal moment-generating function with $n = 3$.

a) Find the moment-generating function of the (joint) marginal distribution of X_1 and X_2

b) Use the moment-generating function to establish that the marginal distribution is multivariate normal with $n = 2$.

c) Find the vector of means μ and the variance-covariance matrix V of this distribution.