MA395 Supplemental Notes on Set Theory

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Cartesian Product

Suppose A and B are sets.

We define the **CARTESIAN PRODUCT** of A and B to be the set of all ordered pairs (a, b) with a being an element of A and b being an element of B.

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The **cartesian product** of A and B is denoted by $A \times B$.

In set builder notation,

$$A \times B = \{(a, b) \ : \ a \in A \text{ and } b \in B\}$$

Cartesian Product Example

Suppose $A = \{1, 2, 3\}$ and $B = \{4, 5\}$.

The **cartesian product** $A \times B$ is:

$$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

Functions

Suppose A and B are sets.

We can use the cartesian product of A and B to define a function f with domain A that takes values in B.

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Recall that the elements of $A \times B$ are ordered pairs (a, b) with $a \in A$ and $b \in B$.

Recall also that a function can be defined by a set of ordered pairs,

$$S \subset A \times B = \{(a, b), (c, d), (e, f), \ldots\}$$

We require only that **every element of A, the domain, appears as the first element of exactly one ordered pair in S.**

Functions

Any subset $S \subset A \times B$ having this property defines, in tabular form, a function

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Note that we do not require every element of B to appear in an ordered pair in S, so B may not be the range of the function f.

However, B will *always* contain the range of f.

Suppose a function $f : A \mapsto B$ is defined by a subset S of $A \times B$. The *range* of f, R_f, is:

$$R_f = \{b \in B \ : \ (a, b) \in S \quad \text{for some } a \in A\}$$

Every element of R_f belongs to B, so

$$R_f \subseteq B$$

One to One Correspondence

A function

$$f : A \mapsto B$$

is called a **one to one correspondence** if:

- The range of f is B
- f is a 1:1 function

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The first condition says the for any $b \in B$, we can find an $a \in A$ such that

$$f(a) = b$$

or, equivalently, if f is defined in tabular form by $S \subset A \times B$, for every $b \in B$ there is some $a \in A$ such that

$$(a, b) \in S$$

One to One Correspondence

The second condition means that no $b \in B$ appears in more than one $(a, b) \in S$.

In other words, no two elements of the domain A are mapped to the same element of the range B.

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Actually, this is the same concept we encountered in calculus when we discussed one to one functions.

One to One Correspondence and Cardinality

The idea of a one to one correspondence provides a way to determine the cardinality of sets.

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In the case of finite sets, it's easy to visualize that the existence of a one to one correspondence between two sets guarantees that they have the same number of elements.

Let's consider two sets, $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$.

The cartesian product $A \times B$ contains 16 ordered pairs,

$$\begin{array}{cccccccc} (a,1) & (a,2) & (a,3) & (a,4) & (b,1) & (b,2) & (b,3) & (b,4) \\ (c,1) & (c,2) & (c,3) & (c,4) & (d,1) & (d,2) & (d,3) & (d,4) \end{array}$$

One to One Correspondence and Cardinality

We can identify the 16 ordered pairs in $A \times B$,

$$A \times B = \left\{ \begin{array}{cccccccc} (a,1) & (a,2) & (a,3) & (a,4) & (b,1) & (b,2) & (b,3) & (b,4) \\ (c,1) & (c,2) & (c,3) & (c,4) & (d,1) & (d,2) & (d,3) & (d,4) \end{array} \right\}$$

with the 16 cells of a 4×4 table,

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & & & & \\ b & & & & \\ c & & & & \\ d & & & & \end{array}$$

One to One Correspondence and Cardinality

As we add elements of $A \times B$ to S, the subset that defines f, we can fill in the table:

$$S = \{(a, 1)\}$$

	1	2	3	4
a	X			
b				
c				
d				

One to One Correspondence and Cardinality

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$$S = \{(a, 1), (b, 3)\}$$

	1	2	3	4
a	X			
b			X	
c				
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One to One Correspondence and Cardinality

As we add elements of $A \times B$ to S, the subset that defines f, we can fill in the table:

$$S = \{(a,1),(b,3),(c,4)\}$$

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a	X			
b			X	
c				X
d				

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c				X
d		X		

The function $f : A \mapsto B$ defined in tabular form by S is in fact a one to one correspondence.

This will be the case any time the table has exactly one entry in each row and in each column.

One to One Correspondence and Cardinality

A bit of thought should convince you of the following:

It is impossible to fill in the table with exactly one entry in each row and column if the number of elements in A and B are not the same:

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
a & X & & & ? \\
b & & X & & ? \\
c & & & X & ?
\end{array}
$$

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	1	2	3	4
a	X			?
b		X		?
c			X	?

	1	2	3
a	X		
b		X	
c			X
d	?	?	?

One to One Correspondence and Cardinality

So, at least for finite sets A and B, the following are equivalent:

- A and B have the same cardinality: $N(A) = N(B)$
- There exists a function $f : A \mapsto B$ that is a one to one correspondence

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- A and B have the same cardinality: $N(A) = N(B)$
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It was George Cantor who extended this idea to infinite sets.

Cardinality - Infinite Sets

Cantor reasoned that, even for infinite sets, the existence of a one to one correspondence between two sets meant that they had the same cardinality.

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For example, it is easy to establish a one to one correspondence between the set

$$S = \{2, 4, 6, 8, \ldots\}$$

and the set of natural numbers

$$\mathcal{N} = \{1, 2, 3, 4, 5, \ldots\}$$

by designating $f(x) = 2x$ for each element of $\mathcal{N}$.

Cardinality - Infinite Sets

The existence of a one to one correspondence between the set

$$S = \{2, 4, 6, 8, \ldots\}$$

and the set of natural numbers

$$\mathcal{N} = \{1, 2, 3, 4, 5, \ldots\}$$

implies that S and $\mathcal{N}$ have the same cardinality (i.e., the same number of elements):

$$N(S) = N(\mathcal{N})$$

Cardinality - Infinite Sets

It took some time for Cantor's ideas to be accepted.

It is difficult at first to accept that

$$S = \{2, 4, 6, 8, \ldots\} \quad \text{and} \quad \mathcal{N} = \{1, 2, 3, 4, 5, \ldots\}$$

have the same number of elements, because clearly S is a proper subset of $\mathcal{N}$:

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In fact, Cantor **defined** an infinite set as a set that can be placed in one to one correspondence with a proper subset of itself.

Cardinality - Infinite Sets

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A set that can be put in one to one correspondence with the natural numbers is called a **countable** set.

The following sets are countable (i.e., have cardinality $\aleph_0$):

- The natural numbers $\mathcal{N}$
- The integers $\mathcal{Z}$
- The rational numbers $\mathcal{Q}$
- The algebraic numbers (roots of polynomials with rational coefficients)

Cardinality - Infinite Sets

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The cardinality of the real numbers is denoted by the symbol $\aleph_1$, and the following relationship can be established:

$$\aleph_1 = 2^{\aleph_0}$$

So, the cardinality of the real numbers $\mathcal{R}$ is the cardinality of the **power set** of the natural numbers $\mathcal{N}$.

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So, the cardinality of the real numbers $\mathcal{R}$ is the cardinality of the **power set** of the natural numbers $\mathcal{N}$.

Remark: Given that the integers, rationals, and algebraics are countable, the fact that the reals are not countable means that the overwhelming majority of real numbers must belong to the remaining category: the transcendentals.

Cantor’s Proof of the Uncountability of the Reals

Cantor established that the real numbers are not countable by the following ingenious argument.

Any countable set can, in principle, be written as a list. Cantor showed that this cannot be done for the real numbers by starting with a list that is claimed to contain all of the real numbers, and proceeding to construct a real number not in the list.

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Recall that a real number can be represented as an integer i followed by a decimal point and an infinite sequence of decimal digits $\{a_1, a_2, a_3, \ldots\}$:

$$i.a_1 a_2 a_3 a_4 \ldots$$

Cantor's Proof of the Uncountability of the Reals

Suppose we are given a list that is claimed to contain all real numbers. The i^{th} entry in the list will consist of an integer i_i followed by a decimal point and an infinite sequence of decimal digits $\{a_{i1}, a_{i2}, a_{i3}, \ldots\}$.

So, the list has the following form:

$$
\begin{array}{c|l}
i & \\
\hline
1 & i_1.a_{11}a_{12}a_{13}a_{14}\ldots \\
2 & i_2.a_{21}a_{22}a_{23}a_{24}\ldots \\
3 & i_3.a_{31}a_{32}a_{33}a_{34}\ldots \\
4 & i_4.a_{41}a_{42}a_{43}a_{44}\ldots \\
5 & i_5.a_{51}a_{52}a_{53}a_{54}\ldots \\
\vdots & \qquad\vdots
\end{array}
$$

Cantor's Proof of the Uncountability of the Reals

$$\begin{array}{c|l} i & \\ \hline 1 & i_1.a_{11}a_{12}a_{13}a_{14}\ldots \\ 2 & i_2.a_{21}a_{22}a_{23}a_{24}\ldots \\ 3 & i_3.a_{31}a_{32}a_{33}a_{34}\ldots \\ 4 & i_4.a_{41}a_{42}a_{43}a_{44}\ldots \\ \vdots & \qquad\quad\vdots \end{array}$$

Cantor's proof constructs a number not in the list by the following algorithm:

- choose the integer part different from i_1
- choose the first decimal digit different from a_{21}
- choose the second decimal digit different from a_{32}
- choose the third decimal digit different from a_{43}

Cantor's Proof of the Uncountability of the Reals

A bit of reflection should convince you that continuing in this fashion will indeed produce at real number that is not in the list, contradicting the claim that the list contains all real numbers.

This type of proof is called *proof by contradiction*.

Some mathematicians advise colleagues and students to avoid proof by contradiction whenever possible. A few even reject the validity of proofs obtained in this fashion.

A more pragmatic approach acknowledges the fact that there are many theorems whose only known proof is by contradiction and accepts these results as valid.