MA395 Set Theory Review

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Definition of a Set

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Sets which meet the criteria that it is always possible to tell whether a given object belongs to the set or not are said to be **well defined**

Examples of Well-Defined Collections

- The set of all students enrolled in MA395 at 8:30 on August 29th, 2006
- The set of all people born on December 12
- Th e set of all people who have walked on the moon

Each of these fits our definition of a set, which incorporates the idea of being well-defined.

Examples of Collections that are not Well-Defined

- The set of all bad people
- The set of all healthy people
- The set of all clouds

Because there is ambiguity as to what objects belong to each of these collections, they are not well defined and therefore do not satisfy our definition of a set.

Definitions

universal set The **universal set** U is the collection of all objects under discussion.

For example, when the topic is real-valued functions of a real variable, the universal set is the set of real numbers.

You can think of the universal set as "the whole universe" (or what is effectively the whole universe in the context of some problem or discussion).

Definitions

compliment of a set If A is a set and U is the universal set, the **compliment** of A, written A^c (or sometimes $\overline{A}$ or $\sim A$), is the set consisting of those elements of U that are not in A. In set builder notation, A^c is:

$$A^c = \{x : \ x \in U \quad \text{and} \quad x \notin A\}$$

The idea of the compliment of a set presupposes that the universal set U has been specified, or is apparent from the context.

Definitions - Union

Union If A and B are sets, the **union** of A and B, written $A \cup B$ (or, equivalently, $B \cup A$), is the set consisting of those elements belonging to either A or B, or both. In set builder notation, $A \cup B$ is:

$$A \cup B = \{x : \ x \in A \quad \text{or} \quad x \in B\}$$

The connective or in this case is interpreted as including the case where both sides are true.

Definitions - Intersection

Intersection If A and B are sets, the **intersection** of A and B, written $A \cap B$ (or, equivalently, $B \cap A$), is the set consisting of those elements belonging to both A or B. In set builder notation, $A \cap B$ is:

$$A \cap B = \{x : \; x \in A \quad \text{and} \quad x \in B\}$$

Definition - Equality of Sets

Equality Two sets A and B are said to be **equal**, written $A = B$, if they have the same elements.

Definitions - Subset, Proper Subset

Subset If A and B are sets, B is said to be a **subseteq** of A if every element of B is also an element of A. That is, $B \subseteq A$ if

$$x \in B \quad \Rightarrow x \in A$$

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Proper Subset If A and B are sets, B is said to be a **proper subset** of A, written $B \subset A$, if B is a subset of A and $B \neq A$.

Definitions - Cardinality

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Example: if $A = \{1, 2, 3, 7, 9\}$, then $N(A) = 5$.

Definitions - Power Set

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Suppose $A = \{1, 2, 3\}$. Then $\mathcal{P}(A)$ is

$$\mathcal{P}(A) = \left\{ \begin{array}{cccc} \{1,2,3\} & \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} & \emptyset \end{array} \right\}$$

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If a set A has cardinality $N(A)$, its power set $\mathcal{P}(A)$ has cardinality $2^{N(A)}$:

$$N\left(\mathcal{P}(A)\right) = 2^{N(A)}$$

Definitions - Algebra of Sets

Algebra of Sets If S is a set, an **algebra of sets over** S, denoted by $\mathfrak{F}(S)$ is a subset of $\mathcal{P}(S)$, the power set of S, that is closed under union, interscetion, and complimentation.

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The elements of $\mathcal{F}(S)$ are **subsets** of S.

The definition implies three considions on the subsets of S that belong to $\mathcal{F}(S)$:

- If A is a subset of S belonging to $\mathcal{F}(S)$, the A^c also belongs to $\mathcal{F}(S)$.
- If A and B are a subsets of S belonging to $\mathcal{F}(S)$, the $A \cup B$ also belongs to $\mathcal{F}(S)$.
- If A and B are a subsets of S belonging to $\mathcal{F}(S)$, the $A \cap B$ also belongs to $\mathcal{F}(S)$.

Definitions - Algebra of Sets

Algebra of Sets Suppose $S = \{1, 2, 3\}$. Then the collection

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Proof: Let $S = \{1, 2, 3\}$ and define

$$\mathcal{F}(S) = \{\{1, 2, 3\}, \emptyset\}$$

We need to show that $\mathcal{F}(S)$ is closed under union, intersection, and complimentation.

Definitions - Algebra of Sets

First, complimentation. We treat S as the universal set and find the compliment of each subset belonging to $\mathcal{F}(S)$:

$$\begin{aligned} \{1,2,3\}^c &= \emptyset \in \mathcal{F}(S) \\ \emptyset^c &= \{1,2,3\} \in \mathcal{F}(S) \end{aligned}$$

Definitions - Algebra of Sets

Next, unions. We need to consider all possible unions of subsets of S belonging to $\mathfrak{F}(S)$, and show that they are all in $\mathfrak{F}(S)$. It's easy in this case because $\mathfrak{F}(S)$ has only two elements, so there is only one union to consider:

$$\{\{1,2,3\} \cup \emptyset\} \quad = \quad \{1,2,3\} \in \mathfrak{F}(S)$$

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$$\{\{1,2,3\} \cup \emptyset\} \quad = \quad \{1,2,3\} \in \mathcal{F}(S)$$

Similarly, there is only one intersection:

$$\{\{1,2,3\} \cap \emptyset\} \quad = \quad \emptyset \in \mathcal{F}(S)$$

This completes the proof that $\mathcal{F}(S)$ is an algebra of sets over S.

$\square$

In general, if S is any set, then $\mathcal{F}(S) = \{S, \emptyset\}$ is an algebra of sets over S.

Definitions - Algebra of Sets

Again let $S = \{1, 2, 3\}$ and this time define

$$\mathcal{F}(S) = \mathcal{P}(S) = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$$

Definitions - Algebra of Sets

Again let $S = \{1, 2, 3\}$ and this time define

$$\mathcal{F}(S) = \mathcal{P}(S) = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\emptyset\}$$

It's more tedious to prove that $\mathcal{F}(S)$ is an algebra of sets over S, but it's done the same way as the previous example.

So, given any set S, we can immediately construct two different algebras of sets over S,

$$\mathcal{F}_1(S) = \{S, \emptyset\}$$

and

$$\mathcal{F}_2(S) = \{\mathcal{P}(S)\}$$