Chapter 4: Special Distributions

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The Uniform Distribution

Perhaps the simplest continuous probability distribution is the uniform distribution.

If X has a uniform distribution (on $[0, 1]$), the pdf of X is:

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Variations can be produced by changing scale and location. A more general version is the uniform density on $[a,b]$ has cdf:

$$F_X(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Exponential Distribution

If Y has an exponential distribution, pdf of Y is

$$f_Y(y) \;=\; \lambda e^{-\lambda y} \quad y \geq 0$$

or, as some authors write it,

$$f_Y(y) \;=\; \frac{e^{-\frac{y}{\lambda}}}{\lambda} \quad y \geq 0$$

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The moment-generating function of the exponential distribution is

$$M_Y(t) \;=\; \frac{\lambda}{\lambda - t}$$

The Weibull Distribution

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Note that when $\beta = 1$, the Weibull reduces to the exponential:

$$f_Y(y) \;=\; \alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta}) \;=\; \alpha x^{0}\exp(-\alpha x)$$

Gamma Distribution

The Gamma distribution is a generalization of the waiting time for the r^{th} event in a Poisson process.

The pdf is:

$$f_Y(y) = \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y}, \quad y > 0$$

Cauchy Distribution

The pdf of the Cauchy distribution is:

$$f_Y(y) = \frac{1}{\pi(1 + x^2)}, \quad x \in (-\infty, \infty)$$

The Cauchy distribution is a rich source of counterexamples. None of the moments exist, and the sampling distribution of the mean of a random sample of size n from a Cauchy distribution is the same as the distribution of a single observation.

Normal Distribution

The normal distribution denoted by $N(\mu, \sigma)$ is a two parameter distribution with density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{y-\mu}{\sigma} \right)^2 \right), \qquad y \in (-\infty, \infty)$$

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An important special case known as the *standard* normal distribution $N(0,1)$ has $\mu = 0$ and $\sigma = 1$. The density function in this case reduces to:

$$f_Z(z) \;=\; \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad z \in (-\infty, \infty)$$

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In the case of the *standard* normal distribution $N(0, 1)$, this reduces to

$$M_Z(z) = e^{\frac{t^2}{2}}$$

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Loosely speaking, the central limit theorem says that the distribution of a sum of independent random variables (with some mild restrictions on their variances) each with mean zero tends to a normal distribution as the number of random variables in the sum becomes large.

Normal Distribution

Remarkably, the statement that the distribution of the sum tends to a normal distribution is true regardless of how the individual random variables are distributed, as long as they are independent and the conditions on their variances are met (roughly, that all of the variances are finite and the bulk of the variation is not concentrated in a few of them).

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The requirement that the means be zero is easily satisfied by subtracting the mean of each individual random variable from the sum.