Chapter 4: Special Distributions

Gene Quinn

Bernouli Trials

Suppose the random variable X takes values 1 and 0 with probabilities p and $q = 1 - p$, respectively. The pdf of X is

$$p_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

Bernouli Trials

Suppose the random variable X takes values 1 and 0 with probabilities p and $q = 1 - p$, respectively. The pdf of X is

$$p_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

The moment-generating function of the Bernoulli distribution is:

$$M_X(t) = \mathsf{E}(e^{tx}) = P(X = 0) \cdot e^{t \cdot 0} + P(X = 1) \cdot e^{t \cdot 1}$$

or

$$M_X(t) = 1 - p + pe^t$$

Binomial Distribution

Perform n independent Bernoulli trials with probability of success p.

The pdf of Y, the sum of the n random variables

$$Y = X_1 + X_2 + \cdots + X_n$$

is given by

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

and is called the **binomial distribution**

Binomial Distribution

Perform n independent Bernoulli trials with probability of success p.

The pdf of Y, the sum of the n random variables

$$Y = X_1 + X_2 + \cdots + X_n$$

is given by

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

and is called the **binomial distribution**

The moment-generating function of the binomial distribution is

$$M_Y(t) = (1 - p + pe^t)^n$$

Generalized Bernouli Trials

Suppose the random variable X takes values 2, 1, and 0 with probabilities p, q, and $1-p-q$, respectively. The pdf of X is

$$p_X(x) = \begin{cases} q & x = 2 \\ p & x = 1 \\ 1-p-q & x = 0 \end{cases}$$

Generalized Bernouli Trials

Suppose the random variable X takes values 2, 1, and 0 with probabilities p, q, and $1-p-q$, respectively. The pdf of X is

$$p_X(x) \; = \; \begin{cases} q & x=2 \\ p & x=1 \\ 1-p-q & x=0 \end{cases}$$

The moment-generating function of the Generalized Bernoulli distribution is:

$$M_X(t) \; = \; \mathsf{E}(e^{tx}) \; = \; P(X=0)\cdot e^{t\cdot 0} + P(X=1)\cdot e^{t\cdot 1} + P(X=2)\cdot e^{t\cdot 2}$$

or

$$M_X(t) \; = \; 1-p-q+pe^{t}+qe^{2t}$$

Extension to more than three possible outcomes is straightforward.

Trinomial Distribution

Perform n independent trials with three outcomes: $\{0, 1, 2\}$ having probabilities p, q, and $1 - p - q$, respectively.

The pdf of Y, the sum of the n random variables

$$Y = X_1 + X_2 + \cdots + X_n$$

is given by

$$p_Y(k_1, k_2) = \frac{n!}{k_1!k_2!(n - k_1 - k_2)!} p^{k_1} q^{k_2} (1 - p - q)^{n - k_1 - k_2}$$

and is called the **trinomial distribution**

Trinomial Distribution

The moment-generating function of the trinomial distribution is

$$M_Y(t) \;=\; (1 - p - q + pe^t + qe^{2t})^n$$

The extension to trials with more than 3 possible outcomes is straightforward. The general case is called the **multinomial** distribution.

Poisson Distribution

Let $n \to \infty$ and $p \to 0$ in such a way that np tends to some positive constant λ.

The pdf of Y is

$$p_Y(k) \ = \ \frac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

and is called the **Poisson distribution**

Poisson Distribution

Let $n \to \infty$ and $p \to 0$ in such a way that np tends to some positive constant λ.

The pdf of Y is

$$p_Y(k) \; = \; \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

and is called the **Poisson distribution**

The moment-generating function of the Poisson distribution is

$$M_Y(t) \; = \; e^{\lambda(e^t - 1)}$$

Geometric Distribution

For a sequence of Bernoulli trials let Y be the number of trials that precede the first success. The pdf of Y is

$$p_Y(k) \; = \; p \cdot (1-p)^{k-1}, \quad k = 1, 2, \ldots$$

and is called the **Geometric distribution**

Geometric Distribution

For a sequence of Bernoulli trials let Y be the number of trials that precede the first success. The pdf of Y is

$$p_Y(k) \;=\; p \cdot (1-p)^{k-1}, \quad k = 1, 2, \ldots$$

and is called the **Geometric distribution**

The moment-generating function of Y is

$$M_Y(t) \;=\; \frac{pe^t}{1-(1-p)e^t}$$

Negative Binomial Distribution

For a sequence of Bernoulli trials let Y be the number of trials that precede the r^{th} success. The pdf of Y is

$$p_Y(k) \;=\; \binom{k-1}{r-1} p^r \cdot (1-p)^{k-r} \quad k = r, r+1, r+2, \ldots$$

and is called the **negative binomial distribution**

Negative Binomial Distribution

For a sequence of Bernoulli trials let Y be the number of trials that precede the r^{th} success. The pdf of Y is

$$p_Y(k) \ = \ \binom{k-1}{r-1} p^r \cdot (1-p)^{k-r} \quad k = r, r+1, r+2, \ldots$$

and is called the **negative binomial distribution**

The moment-generating function of Y is

$$M_Y(t) \ = \ \left(\frac{pe^t}{1-(1-p)e^t} \right)^r$$