Section 3.8: Sums, Products, and Quotients of Independent Random Variables

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Sums of Independent Random Variables

Theorem: (3.8.1 - discrete case) Suppose X and Y are independent discrete random variables with pdfs $p_X(x)$ and $p_Y(y)$, respectively.

Define a random variable W to be the sum of X and Y.

Then the pdf of W is:

$$p_W(w) = \sum_{\text{all } x} p_X(x) \cdot p_Y(w - x)$$

Sums of Independent Random Variables

Example: Suppose two dice are thrown, and the random variables X and Y represent the numbers showing on the faces of the first and second die, respectively, so: $p_X(x) = p_Y(y) = 1/6$.

We are primarily interested in the sum of the two faces, so define a new random variable $W = X + Y$.

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Suppose we want to find $p_W(4)$:

$$p_W(4) = p_X(1) \cdot p_Y(3) + p_X(2) \cdot p_Y(2) + p_X(3) \cdot p_Y(1)$$

$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} \quad = \quad \frac{3}{36}$$

Sums of Independent Random Variables

Theorem: (3.8.1 - continuous case) Suppose X and Y are independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively.

Define a random variable W to be the sum of X and Y.

Then the pdf of W is:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx$$

Sums of Independent Random Variables

Theorem: (3.8.1 - continuous case) Suppose X and Y are independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively.

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The above integral is called the **convolution** of the functions $f_X(x)$ and $f_Y(y)$.

Products of Independent Random Variables

Theorem: (3.8.3) Suppose X and Y are independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively.

Define a random variable W to be the product XY.

Then the pdf of W is:

$$f_W(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(w/x) \cdot f_Y(x)\, dx$$

Quotients of Independent Random Variables

Theorem: (3.8.2) Suppose X and Y are independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$, respectively. Assume that X is zero for at most a set of isolated points.

Define a random variable W to be the quotient Y/X.

Then the pdf of W is:

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) \cdot f_Y(wx)\, dx$$

Sums of Non-Independent Random Variables

Theorem: (variation of 3.8.1 - discrete case) Suppose X and Y are discrete random variables with joint pdf $p_{XY}(x, y)$.

Define a random variable W to be the sum of X and Y.

Then the pdf of W is:

$$p_W(w) = \sum_{\text{all } x} p_{XY}(x, w - x)$$

Sums of Non-Independent Random Variables

Theorem: (variation on 3.8.1 - continuous case) Suppose X and Y are independent continuous random variables with joint pdf $f_{XY}(x, y)$.

Define a random variable W to be the sum of X and Y.

Then the pdf of W is:

$$f_W(w) = \int_{-\infty}^{\infty} f_{XY}(x, w - x)\, dx$$

Products of Non-Independent Random Variables

Theorem: (variation on 3.8.3) Suppose X and Y are independent continuous random variables with joint pdf $f_{XY}(x, y)$.

Define a random variable W to be the product XY.

Then the pdf of W is:

$$f_W(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{XY}(w/x, x)\, dx$$

Quotients of Non-Independent Random Variables

Theorem: (variation of 3.8.2) Suppose X and Y are independent continuous random variables with joint pdf $f_{XY}(x, y)$. Assume that X is zero for at most a set of isolated points.

Define a random variable W to be the quotient Y/X.

Then the pdf of W is:

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_{XY}(x, wx)\, dx$$