Larson and Marx Section 3.7

Gene Quinn

Joint Densities - the Discrete Case

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Y is a random variable witn pdf $f_Y(y)$

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To describe the **joint** behavior of X and Y, we need to introduce the idea of a density function that is a function of both X and Y

Marginal Densities - the Discrete Case

If we know the joint pdf

$$p_{X,Y}(x,y)$$

for two random variables X and Y, we can recover pdf $p_X(x)$ for X by summing over all values of Y,

$$p_X(x) \quad = \quad \sum_{\text{all } y} p_{X,Y}(x,y)$$

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$p_X(x)$ is also said to be the **marginal** density of X.

Marginal Densities - the Discrete Case

Example: The discrete distribution with probability density function

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

is called the *poisson distribution*.

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This is a probability density function by virtue of the fact that the series expansion of e^λ is:

$$e^\lambda = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots$$

Marginal Densities - the Discrete Case

Adding up the probabilities $p_X(x)$ for $x = 0, 1, 2, \ldots$ gives:

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$$= e^{-\lambda} \left(e^{\lambda} \right) = 1$$

Marginal Densities - the Discrete Case

The **bivariate** poisson distribution is defined by the joint pdf

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If we sum over all values of Y, we obtain the marginal pdf of X,

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We have shown that the sum in parentheses is 1, so the marginal pdf of X is

$$p_X(x) = e^{-\lambda_1} \frac{\lambda_1^x}{x!}$$

which is a (univariate) poisson.

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Definition: Two random variables defined on the same set of real numbers are said to be **jointly continuous** if there exists a function

$$f_{X,Y}(x, y)$$

such that for any region R in the xy-plane,

$$P((X, Y) \in R) = \int\int_R f_{X,Y}(x, y)\, dx\, dy$$

$f_{X,Y}(x, y)$ is called the **joint pdf** of X and Y.

Joint Densities - the Continuous Case

Suppose two random variables X and Y have the joint density function

$$f_{X,Y}(x,y) = 1, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq \frac{1}{2}$$

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This is a special case of the *joint uniform* pdf.

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For any region R belonging to the rectangle with width 2 and height $\frac{1}{2}$, the probability that (X, Y) belongs to R is just the area of R.

Marginal Densities - the Continuous Case

As in the discrete case, if we have a joint pdf for two continuous random variables X and Y,

$$f_{X,Y}(x,y)$$

we can obtain univariate or *marginal* pdfs for X and Y.

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As in the discrete case, if we have a joint pdf for two continuous random variables X and Y,

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we can obtain univariate or *marginal* pdfs for X and Y.

Theorem: (3.7.2) Suppose X and Y are jointly continuous with joint pdf $f_{X,Y}(x, y)$. Then the **marginal pdfs** of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

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The marginal pdf $f_X(x)$ is given by

$$f_X(x) = \int_0^{1/2} 1\,dy \quad = \quad y\Big]_0^{1/2} = \frac{1}{2}, \quad 0 \le x \le 2$$

Joint Densities - the Continuous Case

The marginal pdf $f_Y(y)$ is given by

$$f_Y(y) = \int_0^2 1\,dx \quad = \quad x]_0^2 = 2, \quad 0 \leq y \leq 1/2$$

Joint CDFs

Definition: Let X and Y be random variables. The **joint cumulative distribution function** or **joint cdf** of X and Y is

$$F_{X,Y}(u,v) \quad = \quad P(\ X \leq u \quad \text{and} \quad Y \leq v)$$

Obtaining Joint PDFs From Joint CDFs

Theorem: (3.7.3) Let X and Y be random variables with joint cdf

$$F_{XY}(u,v)$$

Then the joint pdf of X and Y,

$$f_{XY}(x,y).$$

is given by the second partial derivative

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

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(provided $F_{X,Y}(x,y)$ has continuous second partial derivatives)

Obtaining Joint PDFs From Joint CDFs

Example: Let X and Y be random variables with joint cdf

$$F_{XY}(u,v) = \frac{1}{2}\left(u^3v^2 + u^2v^3\right)$$

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Recall that we evaluate the mixed second order partial derivative in two steps:

$$\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F_{XY}(x,y)\right)$$

Obtaining Joint PDFs From Joint CDFs

The first step is to take the partial derivative of $F_{XY}(x, y)$ with respect to x

$$\frac{\partial}{\partial x}\frac{1}{2}\left(x^3y^2 + x^2y^3\right) = \frac{1}{2}\left(3x^2y^2 + 2xy^3\right)$$

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The second step is to take the partial derivative of the result with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}\left(3x^2y^2+2xy^3\right)\right) = \frac{1}{2}\left(6x^2y+6xy^2\right)$$

so

$$f_{XY}(x,y) = 3\left(x^2y+xy^2\right)$$

Multivariate Densities

The bivariate versions of definitions and theorems in this section extend in the obvious way to the case of more than two variables.

Definition: The **joint pdf** of a set of n discrete random variables

$$X_1,\, X_2,\, X_3,\, \ldots X_n$$

is defined to be:

$$p_{X_1 \cdots x_N}(x_1, x_2, x_3, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, X_3 = x_3, \ldots, X_n = x_n)$$

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In the case of n continuous random variables, the **joint pdf** is the function that, for any region R in $\mathcal{R}^n$, satisfies

$$P\left((X_1, X_2, \ldots, X_n) \in R\right) = \int \cdots \int_R f_{X_1 \cdots X_n}(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n$$

Multivariate Marginal Densities

In the case of the joint pdf of 2 random variables

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we have 2 marginal pdfs,

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- and so on until $\binom{n}{n-1}$ $n-1$-variate marginal pdfs

Multivariate Marginal Densities

In each case, the marginal pdf is obtained by integrating some subset of the n random variables over its support.

For example, the marginal pdf of X_2 would be:

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \cdots X_n}(x_1, \ldots, x_n) dx_1 \, dx_3 \cdots dx_n$$

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The marginal pdf of X_2 and X_4 would be the joint pdf:

$$f_{X_2 X_4}(x_2, x_4) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \cdots X_n}(x_1, \ldots, x_n) dx_1\, dx_3\, dx_5 \cdots dx_n$$

Independence of Two Random Variables

Definition: Two random variables X and Y are said to be **independent** if for every pair of intervals A and B,

$$P(X \in A \text{ and } Y \in B) \;=\; P(X \in A) \cdot P(Y \in B)$$

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Definition: Two random variables X and Y are said to be **independent** if for every pair of intervals A and B,

$$P(X \in A \text{ and } Y \in B) = P(X \in A) \cdot P(Y \in B)$$

The above definition suggests that, for a pair of independent random variables, the joint pdf might factor into the product of the marginals.

The following important theorem asserts this fact.

Independence of Two Random Variables

Theorem: Two random variables X and Y with joint pdf

$$f_{XY}(x, y)$$

are independent if and only if

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

where:

- $f_X(x)$ is the marginal pdf of X
- $f_Y(y)$ is the marginal pdf of Y

Independence of Two Random Variables

Another way of looking at independence of two random variables is the following:

If X and Y are independent,

- knowledge of X provides no information about Y
- knowledge of Y provides no information about X

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In general, if X and Y are **not** independent, knowledge of one tells us something about the other.

Independence of More Than Two Random Variables

The concept of independence of two random variables extends to the case of $n > 2$ random variables:

Definition: The random variables

$$X_1 \; X_2, \ldots X_n$$

are said to be **independent** if, for every

$$(x_1, x_2, \ldots, x_n) \in \mathcal{R}^n,$$

$$f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) \;=\; f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

where $f_{X_i}(x_i)$ is the marginal pdf of x_i, $i \in \{1, 2, \ldots, n\}$.

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An important special case of n independent random variables occurs when the marginal pdfs of the X_i are identical.

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all having the same pdf.

In this case, the joint pdf is:

$$f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

where $f_X(x)$ is common pdf of the x_i.