Larson and Marx Section 3.4

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Continuous Random Variables

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So far, we have restricted our attention to the case where S is finite or countably infinite.

Problems where this is not the case arise often, so we need to extend our definitions accordingly.

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Fortunately, this is neither necessary nor desirable.

Interval Functions

One way around the problem is to consider only a certain class of subsets when assigning probabilities.

For example, we can consider only subsets that qualify as **intervals**:

For $a, b \in \mathcal{R}$, with $a < b$, these are sets of the form:

$$(a, b) \qquad I = \{x \ : \ a < x < b\}$$

$$[a, b) \qquad I = \{x \ : \ a \leq x < b\}$$

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We also allow the limiting cases where one or both of a and b are $\pm\infty$,

$$(a, \infty), \ (-\infty, b], \ , (-\infty, \infty), \quad \text{etc.}$$

Continuous Random Variables

Now consider the experiment:

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We will define the continuous random variable X to be simply the number selected, so the function defining our random variable is:

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according to the rule:

$$X(x_i) = x_i$$

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A continuous random variable is often an identity function, and in this regard the continuous case is simpler than the discrete case.

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We will use the idea of an interval function in the following definitions:

Definition: (3.4.2) A function X that maps a subset of the real numbers into the real numbers is called a **continuous random variable**.

The **probability density function** (pdf) of X is defined to be the function

$$f_X(x)$$

having the property that for any numbers a and b,

$$P(a \leq X \leq b) \quad = \quad \int_a^b f_X(x)dx$$

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If we assume that every number in the interval is equally likely to be selected, a reasonable **probability density function** is:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

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A probability density function is said to have **support** on the subset of $\mathcal{R}$ where it is positive.

The above probability density function has support on $[0, 1]$.

The Uniform Distribution

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This is true for our density function,

$$\int_0^1 1dx = x]_0^1 = 1$$

and in fact the probability distrubution associated with this density function is known as the **uniform distribution**.

The Uniform Distribution

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So, if X has the uniform distribution, the probability that $a \leq X \leq b$ is just the length of the interval $[a, b]$.

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Continuous Cumulative Distribution Functions

Definition: (3.4.3) The **cumulative distribution function** (cdf) of a continuous random variable X is defined to be

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Note that outside of its support, the integral of the density function is zero.

So, while stating the lower limit of integration as $-\infty$ produces the most general answer, in practice we can ignore values outside the support of $f_X(x)$.

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Since $f_X(x)$ only has support on $[0, 1]$, we can ignore the rest of the real line, so

$$F_X(x) = \int_{0}^{x} 1dr \quad = \quad r]_0^x = x = P(X \leq x)$$

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It follows immediately from the Fundamental Theorem of Calculus that:

Theorem: (3.4.1) Let $F_X(x)$ be the cdf of a continuous random variable X. Then

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That is, the derivative of a continuous cumulative distribution function is the corresponding probability density function.

Theorems involving the cdf

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$$d) \quad \lim_{x \to -\infty} F_X(s) = 0$$

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Suppose that Y is a random variable related to X by

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Then the probability density function of Y is

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

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Suppose instead we are choosing a real number at random from the interval $[3, 7]$.

Most calculators would not have this function built in, but we can produce the same result by drawing a number randomly from $[0, 1]$, multiplying it by 4, and adding 3:

$$Y = 4X + 3$$

Linear Transforms of Continuous Random Variables

Theorem 3.4.3 says that the probability density function of $Y = 4X + 3$ will be given by:

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For these values, $f_X(x) = 1$, so by substitution $f_Y(y)$ reduces to:

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Generalizations of the uniform distribution that arise in this way are called a **rectangular** distributions.

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The cdf for this probability distribution is

$$F_X(x) = \int_0^x f_X(r) dr = \left.\frac{2r^2}{2}\right]_0^x \quad = \quad x^2, \quad 0 \leq x \leq 1$$

Other Examples of pdfs

The probability distribution with pdf

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$$\int_0^t \frac{1}{\mu} e^{-x/\mu} dx \quad = \quad -\frac{\mu e^{-x/\mu}}{\mu}\Bigg]_0^t \quad = \quad 1 - e^{-t/\mu}$$

As $t \to \infty$, the integral approaches 1.

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The cdf is

$$F_X(x) = \int_0^x \frac{1}{\mu} e^{-r/\mu} dr \quad = \quad 1 - e^{-x/\mu}, \quad 0 \le x$$

Failure Analysis Example

Suppose the time to failure of a light bulb has an exponential distribution with $\mu = 500$ (hours). Then the pdf of the time to failure is:

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What is the probability that a given light bulb lasts less than 400 hours?

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The cdf is

$$F_X(x) = \quad = \quad 1 - e^{-x/500}, \quad 0 \le x$$

so

$$P(X \le 400) = F_X(400) = 1 - e^{-400/500} = 0.551$$

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What is the probability that a given light bulb lasts more than 600 hours?

The probability that it lasts 600 hours or less is $F_X(600)$, so the probability that it lasts longer than 600 hours is

$$1 - P(X \leq 600) = 1 - F_X(600) = 1 - (1 - e^{-600/500}) = 0.301$$