Section 3.12: Moment-Generating Functions

Gene Quinn

Moment-Generating Functions: the Discrete Case

Definition: (3.12.1) Let X be a discrete random variable. The **moment-generating function** or mgf for X is defined to be:

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } k} e^{tk} p_X(k)$$

at all values of t for which the expected value exists.

Moment-Generating Functions: the Discrete Case

Definition: (3.12.1) Let X be a discrete random variable. The **moment-generating function** or mgf for X is defined to be:

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } k} e^{tk} p_X(k)$$

at all values of t for which the expected value exists.

The variable t is essentially a dummy variable that has no physical interpretation.

Moment-Generating Functions: the Continuous Case

Definition: (3.12.1) Let X be a continuous random variable. The **moment-generating function** or mgf for X is defined to be:

$$M_X(t) \;=\; E(e^{tX}) \;=\; \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

at all values of t for which the expected value exists.

Finding Moments

Theorem: (3.12.1) Let W be a random variable with pdf $f_W(w)$, and let $M_W(t)$ be the moment-generating function for W. Then the r^{th} moment of W about the origin is given by the r^{th} derivative of $M_W(t)$ evaluated at $t = 0$:

$$M_W^{(r)}(0) = E(W^r)$$

provided the r^{th} moment exists.

Finding Moments

Theorem: (3.12.1) Let W be a random variable with pdf $f_W(w)$, and let $M_W(t)$ be the moment-generating function for W. Then the r^{th} moment of W about the origin is given by the r^{th} derivative of $M_W(t)$ evaluated at $t = 0$:

$$M_W^{(r)}(0) \;=\; E(W^r)$$

provided the r^{th} moment exists.

IF W is continuous, the pdf $f_W(w)$ must be sufficiently smooth to allow the order of integration and differentiation to be interchanged in the expression

$$M_W^{(1)}(0) \;=\; \frac{d^r}{dw^r}\int_{-\infty}^{\infty} e^{tw} f_W(w)\,dw \;=\; \int_{-\infty}^{\infty} \frac{d^r}{dw^r} e^{tw} f_W(w)\,dw$$

Finding Moments

Example: Find the moment-generating function of a random variable with the exponential distribution.

Finding Moments

Example: Find the moment-generating function of a random variable with the exponential distribution.

From the definition,

$$M_Y(y) \;=\; E(e^{ty} \;=\; \int_0^\infty e^{ty} \cdot \lambda e^{-\lambda y}\, dy$$

Finding Moments

Example: Find the moment-generating function of a random variable with the exponential distribution.

From the definition,

$$M_Y(y) \;=\; E(e^{ty} \;=\; \int_0^\infty e^{ty} \cdot \lambda e^{-\lambda y}\, dy$$

$$M_Y(y) \;=\; \int_0^\infty \lambda e^{(t-\lambda)y}\, dy \;=\; \frac{\lambda}{\lambda - t}$$

Identifying pdf's

Theorem: 3.12.2 Suppose W_1 and W_2 are random variables for which

$$M_{W_1}(t) = M_{W_2}(t)$$

for values of t in some interval that contains zero. Then:

$$f_{W_1}(w) = f_{W_2}(w)$$

Identifying pdf's

Theorem: 3.12.2 Suppose W_1 and W_2 are random variables for which

$$M_{W_1}(t) \; = \; M_{W_2}(t)$$

for values of t in some interval that contains zero. Then:

$$f_{W_1}(w) \; = \; f_{W_2}(w)$$

This means that if W_1 and W_2 are random variables with identical mgfs, then they must have the same pdf.

Identifying pdf's

Theorem: 3.12.3a Suppose W is a random variable with moment-generating function $M_W(t)$, and

$$V = aW + b$$

Then the moment-generating function of V is:

$$M_V(t) = e^{bt} M_W(at)$$

Identifying pdf's

Theorem: 3.12.3a Suppose W is a random variable with moment-generating function $M_W(t)$, and

$$V = aW + b$$

Then the moment-generating function of V is:

$$M_V(t) = e^{bt} M_W(at)$$

This theorem gives the moment-generating function for a linear function of a random variable in terms of the moment-generating function of the original variable.

Identifying pdf's

Theorem: 3.12.3b Suppose

$$W_1, W_2, \ldots, W_n$$

are independent random variables with respective moment-generating functions

$$M_{W_1}(t), M_{W_2}(t), \ldots, M_{W_n}(t)$$

Identifying pdf's

Theorem: 3.12.3b Suppose

$$W_1, W_2, \ldots, W_n$$

are independent random variables with respective moment-generating functions

$$M_{W_1}(t), M_{W_2}(t), \ldots, M_{W_n}(t)$$

Suppose the random variable W is the sum

$$W = W_1 + W_2 + \cdots + W_n$$

Then the moment-generating function of the sum W is

$$M_W(t) = M_{W_1}(t) \cdot M_{W_2}(t) \cdots M_{W_n}(t)$$

Identifying pdf's

Theorem: 3.12.3b Suppose

$$W_1, W_2, \ldots, W_n$$

are independent random variables with respective moment-generating functions

$$M_{W_1}(t), M_{W_2}(t), \ldots, M_{W_n}(t)$$

Suppose the random variable W is the sum

$$W = W_1 + W_2 + \cdots + W_n$$

Then the moment-generating function of the sum W is

$$M_W(t) = M_{W_1}(t) \cdot M_{W_2}(t) \cdots M_{W_n}(t)$$

This theorem provides a way to find the mgf of a sum of independent random variables.