Combinatorics

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The Multiplication Rule

Multiplication Rule: If operation A can be performed m different ways, and operation B can be performed n different ways, then the *sequence*

$$(\text{operation } A, \text{ operation } B)$$

can be performed in $m \cdot n$ different ways.

Corollary to the Multiplication Rule

Corollary: If for some positive integer k the operations

$$\{A_1, A_2, \ldots A_k\}$$

can be performed in

$$\{n_1, n_2, \ldots n_k\}$$

ways, respectively, then the ordered sequence

$$(\text{operation } A_1, \text{operation } A_2, \ldots, \text{operation } A_k)$$

can be performed in

$$n_1 \cdot n_2 \cdots n_k$$

ways.

Permutations

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$$_nP_k$$

Theorem: (2.6.1) The number of permutations of length k that can be formed from a set of n distinct elements, with repetitions not allowed, is

$$_nP_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Permutations of Non-distinct Objects

Theorem: (2.6.2) The number of ways to arrange n objects with n_1 being of kind 1, n_2 being of kind 2,. . ., and n_k being of kind k is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

where

$$\sum_i = 1^n n_i = n$$

Combinations

If we choose k of n distinct objects with order being important, we have seen that there are

$$ {}_nP_k = \frac{n!}{(n-k)!} \quad \text{permutations} $$

In many applications, order is not important.

For example, if we want to know the number of poker hands, it should be the number of 5 member subsets of a standard deck of 52 cards.

In this case, the order in which the cards are drawn is irrelevant.

Combinations

There is an easy way to determine this number once we know that there are

$$ {}_{52}P_5 = \frac{52!}{(52-5)!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 $$

permutations with 5 cards.

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permutations with 5 cards.

Since each subset of 5 cards can be arranged in

$$_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!$$

ways, to get the number of subsets, we should divide the number of permutations by $5!$ to get:

$$\frac{52!}{5!}(52-5)!$$

Combinations

This construct

$$\frac{52!}{5!(52-5)!}$$

occurs so often it is given two symbols:

$${}_{52}C_5 \quad \text{and} \quad \binom{52}{5}$$

and referred to as the number of **combinations** of 52 things taken 5 at a time.

Combinations

Theorem: (2.6.3) The number of ways to form combinations of size k from a set of n distinct objects, with no repetitions, is given by

$$ {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} $$

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Combinations

The symbol

$$\binom{n}{k}$$

is also read as:

n things taken k at a time

or

n choose k

Poker Hands

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This is given by:

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$$= \frac{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) \cdot 47!}{5! \cdot 47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

Poker Hands

The factors can be rearranged to produce:

$$= \left(\frac{52}{4}\right)\left(\frac{51}{3}\right)\left(\frac{50}{5}\right)\left(\frac{49}{1}\right)\left(\frac{48}{2}\right)$$

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$$= 2,598,960$$

Bridge Hands

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After some computation, this reduces to

$$= 635,013,559,600$$

Lottery Tickets

Suppose a lottery game consists of choosing 6 numbers from the set

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How many different choices can be made?

The answer is 36 choose 6, or

$$\binom{36}{6} = \frac{36!}{6!(36-6)!}$$

$$= \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \quad = \quad 6 \cdot 7 \cdot 17 \cdot 11 \cdot 8 \cdot 31 \quad = \quad 1,947,792$$

Simple Random Samples

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A sampling scheme is called a **simple random sample** if each of the possible samples has an equal chance of being selected.

Binomial Coefficients

When a binomial expression

$$(a+b)$$

is raised to a power n, the result is a sum of $n+1$ terms

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Binomial Coefficients

Written as a summation,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} a^0 b^n$$

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Answer: In the expansion of $(1+x)^n$, for any $k \in \{0, 1, 2, \ldots, n\}$, the coefficient of x^k is

$$\binom{n}{k}$$

So, the coefficient of x^{30} is

$$\binom{35}{30} = \frac{35!}{30! \cdot 5!} = \frac{35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 7 \cdot 17 \cdot 11 \cdot 8 \cdot 31 \quad = \quad 324,632$$

Binomial Coefficients

For the special case $a = b = 1$,

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This produces the identity:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$