Calculus Review: Multiple Integrals

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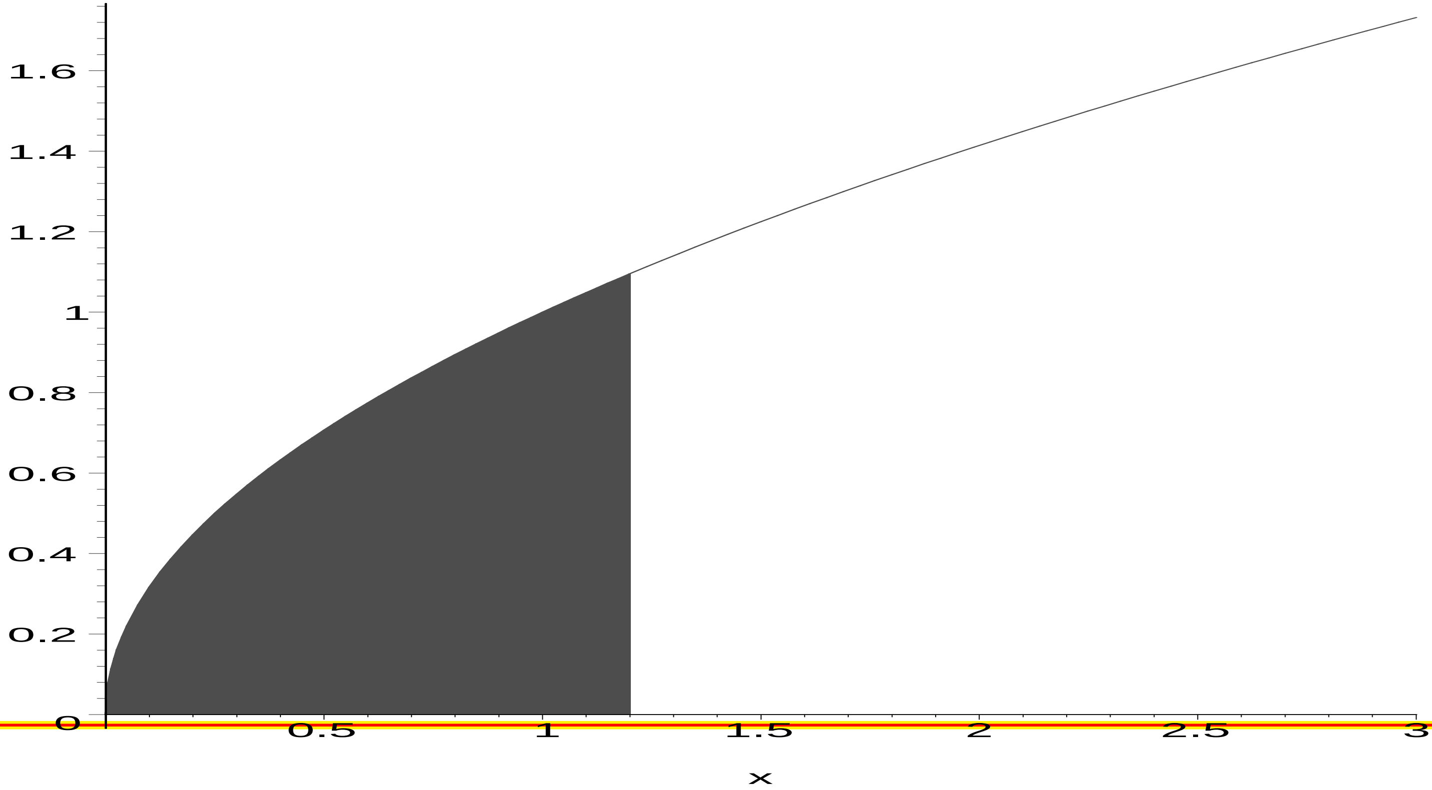
The Univariate Case

The definite integral of a function $y = f(x)$ involves an interval $[a, b]$ on the real line. For example,

$$\int_0^{1.2} \sqrt{x}\, dx$$



The Bivariate Case

Many applications involve functions of more than one variable, so we need to generalize the integration process to cover functions of this type.

We start with a real-valued function of two variables, x and y,

$$f \;:\; (x, y) \mapsto \mathbb{R}$$

The Bivariate Case

The simplest generalization of the definite integral over an interval is the **double integral** over a **rectangle** R,

$$R = \{(x, y) : a \leq x \leq b \quad \text{and} \quad c \leq y \leq d$$

denoted by:

$$\iint_R f(x, y)\, dA$$

where dA represents an element of the area of the rectangle.

The Bivariate Case - Example

Suppose a bivariate function $f(x, y)$ is defined by:

$$f(x, y) \quad = \quad 1$$

on the rectangle

$$R \;=\; \{(x, y) \;:\; 0 \leq x \leq 2 \quad \text{and} \quad 0 \leq y \leq 1/2\,\}$$

The Bivariate Case - Example

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The double integral over the rectangle R

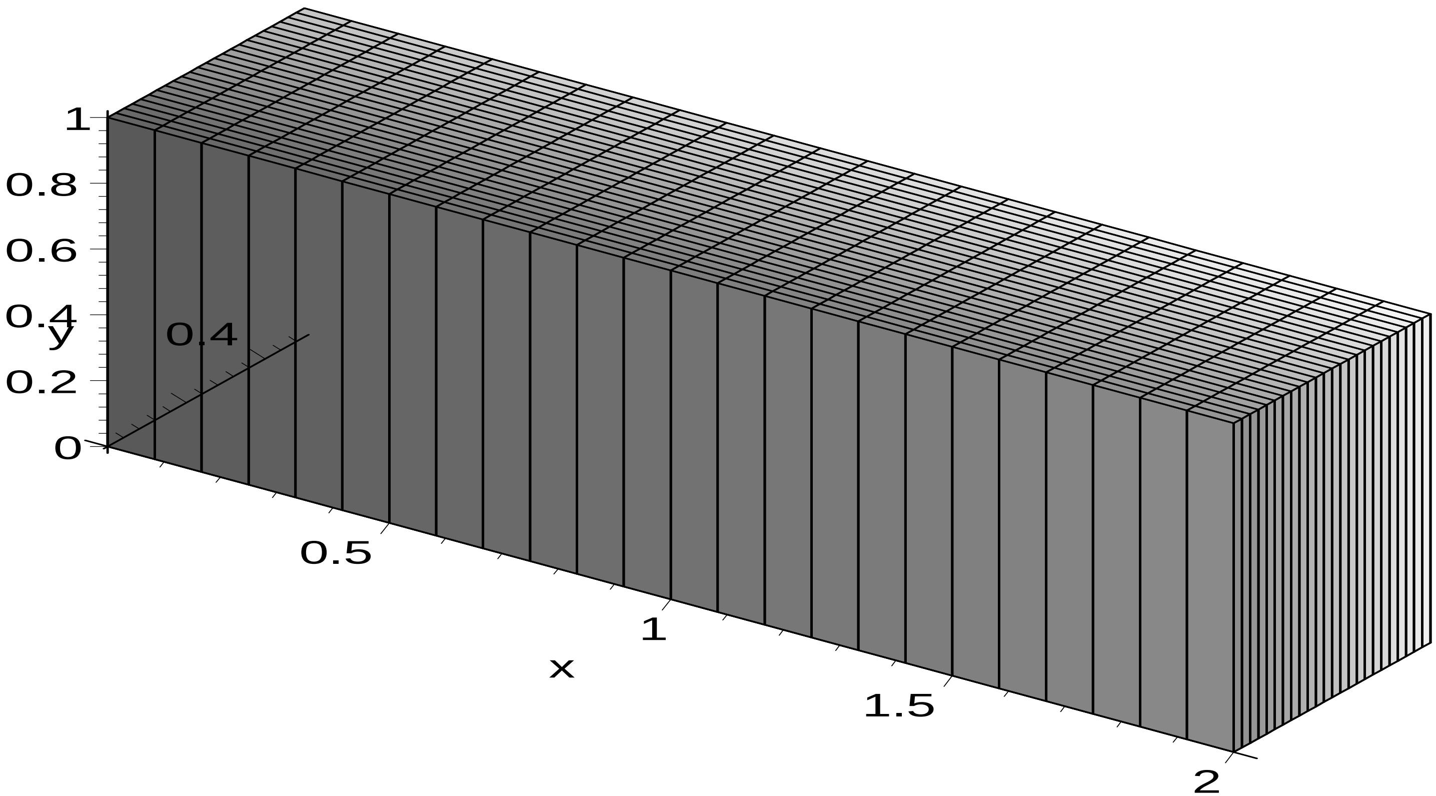
$$\iint_R f(x, y)\, dA \quad = \quad \int_0^2 \int_0^{\frac{1}{2}} 1 \, dy \, dx$$

represents a **volume** in three dimensions.

The Bivariate Case - Example

In this case, the double integral represents the volume of a parallelepiped

$$\int_0^2 \int_0^{\frac{1}{2}} 1 \, dx \, dy$$



Iterated Integrals - Fubini's Theorem

Fubini's Theorem addresses the problem of evaluating a double integral over a rectangle in the xy-plane.

Suppose $f(x, y)$ is defined on the rectangle

$$R = \{(x, y) : a \leq x \leq b \quad \text{and} \quad c \leq y \leq d\}$$

Iterated Integrals - Fubini's Theorem

Fubini's theorem states that we can evaluate the double integral as two successive single integrals,

$$\iint_R f(x,y)\,dA \quad = \quad \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

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and that the order of integration is not important:

$$\iint_R f(x,y)\,dA \quad = \quad \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

The Bivariate Case - Example

In our example, our parallelipiped has dimensions:

$$\text{width=}2 \quad \text{length=}\frac{1}{2} \quad \text{height=}1$$

so we can easily compute the volume as:

$$V = 2 \times \frac{1}{2} \times 1 \quad = \quad 1$$

The Bivariate Case - Example

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so we can easily compute the volume as:

$$V = 2 \times \frac{1}{2} \times 1 \quad = \quad 1$$

We can also compute the volume using Fubini's theorem,

$$V = \int_0^2 \left(\int_0^{\frac{1}{2}} 1 \, dy \right) dx$$

The Bivariate Case - Example

To find

$$V = \int_0^2 \left(\int_0^{\frac{1}{2}} 1\, dy \right) dx$$

we first evaluate the inner integral,

$$V = \int_0^2 \left(y|_0^{\frac{1}{2}} \right) dx$$

The Bivariate Case - Example

and obtain

$$V = \int_0^2 \frac{1}{2}\,dx$$

The Bivariate Case - Example

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$$V = \int_0^2 \frac{1}{2}\, dx$$

Now the second integration gives

$$V = \int_0^2 \frac{1}{2} dx = \frac{1}{2} x \Big|_0^2 = 1$$