1. Chebychev's Inequality (discrete random variables)

The following result is of great importance:

Suppose Y is a discrete random variable and $V(Y)$ exists.

Suppose also that $E(Y) = 0$. Then for any $a > 0$,

$$P(|Y| \geq a) \leq \frac{E(Y^2)}{a^2}$$

Proof. By hypothesis, Y is a discrete random variable with $E(Y) = 0$ and $V(Y)$ exists.

By definition,

$$V(Y) = E(Y^2) - [E(Y)]^2$$

so the existence of $V(Y)$ and $E(Y)$ together imply the existence of $E(Y^2)$, and the above expression reduces to

$$V(Y) = E(Y^2)$$

By definition,

$$E(Y^2) = \sum_y y^2 \cdot p(y)$$

Now for each value of y, either $|y| < a$ or $|y| \geq a$, so we can write $E(Y^2)$ as two sums:

$$E(Y^2) = \sum_{|y|<a} y^2 \cdot p(y) + \sum_{|y|\geq a} y^2 \cdot p(y)$$

Both factors in the summands are nonnegative, so the first sum must be greater than or equal to zero, and we can convert to an inequality and eliminate it:

$$E(Y^2) \geq \sum_{|y|\geq a} y^2 \cdot p(y)$$

For the second sum $|y| \geq a$ in all cases, so $y^2 \geq a^2$ and therefore

$$E(Y^2) \geq \sum_{|y|\geq a} y^2 \cdot p(y) \geq \sum_{|y|\geq a} a^2 \cdot p(y)$$

but

$$\sum_{|y|\geq a} a^2 \cdot p(y) = a^2 \sum_{|y|\geq a} p(y) = a^2 \cdot P(|Y| \geq a)$$

so finally

$$E(Y^2) \geq a^2 \cdot P(|Y| \geq a)$$

and we can write this as

$$P(|Y| \geq a) \leq \frac{E(Y^2)}{a^2}$$

$\square$