1. Preliminaries

The setting is a vector of n independent, identically distributed (IID) random variables with finite variance (which implies finite expectation).

Without loss of generality, we will assume that the mean is zero and the variance is one. This can be justified by noting that if Y is a random variable with

$$E(Y) = \mu \quad \text{and} \quad V(Y) = \sigma^2 < \infty$$

then the transformed random variable

$$Z = \frac{Y - \mu}{\sigma} \quad \text{has} \quad E(Z) = 0 \quad \text{and} \quad V(Z) = 1 = E(Z^2)$$

Recall that if Z has moment-generating function $m_Z(t)$, then aZ has moment-generating function $m_Z(at)$. Using this result, for a nonnegative constant n, the moment-generating function of $Z/\sqrt{n}$ is:

$$m_{Z/\sqrt{n}}(t) = m_Z\left(\frac{t}{\sqrt{n}}\right)$$

Recall also that every moment-generating function $m_Y(t)$ has the following properties:

$$m(0) = 1, \quad m'(0) = E(Y), \quad m''(0) = E(Y^2)$$

Finally, remember from calculus that if a function f has a power series expansion centered at zero, it is given by the formula:

$$f(t) \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \quad = \quad f(0) + f'(0)t + \frac{f''(0)}{2!} t^2 + \cdots$$

The above power series is called a *Maclaurin series* or, more generally, a *Taylor series*. Truncated power series play an important role in applied Mathematics.

The difference between the full series and the truncated series is called the *remainder*, so if we keep say, the first two terms (i.e., terms up to index $n = 1$), the remainder is:

$$R_1(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n - \sum_{n=0}^{1} \frac{f^{(n)}(0)}{n!} t^n \quad = \quad \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$$

The *Lagrange form* of the remainder is a theorem that states that if we keep the first k terms of the power series, the remainder can be expressed in the following form:

$$R_k(t) \quad = \quad \sum_{n=k}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \quad = \quad \frac{f^{(k+1)}(\xi)}{(k+1)!} t^{(k+1)} \quad \text{for some} \quad \xi \in (0,t)$$

so in the case $n = 1$, we can write the following exact expression for $f(t)$:

$$f(t) = f(0) + f'(0)t + \frac{f''(\xi)}{2} t^2 \quad \text{for some} \quad \xi \in (0,t)$$

Finally, we will need a result from calculus that says that:

$$\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

2. Proof of the Central Limit Theorem

Theorem (Central Limit Theorem)**.** *Suppose $(Z_1, Z_2, \ldots, Z_n)$ are independent, identically distributed random variables with expected value $E(Z_i) = 0$ and variance $V(Y_i) = 1$, and define*

$$U_n \quad = \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots$$

Then the distribution function of U_n converges to that of the standard normal distribution, $N(0,1)$, as $n \to \infty$.

Proof. Let $m_Z(t)$ be the moment-generating function of each Z_i. Using Taylor's theorem with remainder, we can write the following (exact) expression for $m_Z(t)$ as a Taylor series with two terms and the remainder:

$$m_Z(t) = m_Z(0) + m_Z'(0) \cdot t + m_z''(\xi) \cdot \frac{t^2}{2} \quad \text{for some} \quad 0 < \xi < t$$

By the properties of moment-generating functions, $m(0) = 1$ and $m'(0) = E(Z)$ which is zero in this case, so our expression becomes

$$m_Z(t) = 1 + m_z''(\xi) \cdot \frac{t^2}{2} \quad \text{for some} \quad 0 < \xi < t$$

Now

$$U_n \quad = \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \quad = \quad \sum_{i=1}^{n} \frac{Z_i}{\sqrt{n}}, \quad n = 1, 2, \ldots$$

so again using the properties of moment-generating functions and the fact that the Z_i are independent, we can write the moment-generating function for U_n as:

$$m_{U_n}(t) = \prod_{i=1}^{n} m_{Z_i}\left(\frac{t}{\sqrt{n}}\right) = \left[m_Z\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

replacing

$$m_Z\left(\frac{t}{\sqrt{n}}\right)$$

with the Taylor series version

$$1 + \frac{m_Z''(\xi)}{2} \cdot \left(\frac{t}{\sqrt{n}}\right)^2 \quad \text{for some} \quad \xi_n \in (0, t/\sqrt{n})$$

we obtain

$$m_{U_n}(t) \quad = \quad \left[1 + \frac{m_Z''(\xi_n)}{2}\left(\frac{t}{\sqrt{n}}\right)^2\right]^n \quad \text{for some} \quad \xi_n \in (0, t/\sqrt{n})$$

which can be rearranged to

$$m_{U_n}(t) \quad = \quad \left[1 + \frac{m_Z''(\xi_n)(t^2/2)}{n}\right]^n \quad \text{for some} \quad \xi_n \in (0, t/\sqrt{n})$$

Noting that as $n \to \infty$, $\xi_n \to 0$ (because $\xi_n \in (0, t/\sqrt{n})$, so

$$\lim_{n\to\infty} m_Z''(\xi_n) \quad = \quad m_Z''(0) = E(Z^2) = 1$$

and we can now write

$$\lim_{n\to\infty} m_{U_n}(t) = \lim_{n\to\infty}\left[1 + \frac{t^2/2}{n}\right]^n = e^{t^2/2}$$

which is the moment-generating function of a standard normal variate.

$\square$