MA395 Takehome Quiz 6

Name:

1) (heads) (Problem 3.11.2) Suppose a die is rolled 6 times. Let X be the total number of 4's that occur and let Y be the number of 4's in the first two tosses. Find $P_{Y|x}(y)$.

Solution: The event ($X = x$ and $Y = y$) occurs when we roll y 4's in the first two tosses, and $x - y$ in the last 4.

We may consider the first two tosses as a binomial experiment with $n = 2$ trials and $p = 1/6$, and the probability of y successes is

$$P(Y = y) \ = \ \binom{2}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{2-y}, \quad 0 \le y \le 2$$

Likewise, we may consider the last four tosses as a binomial experiment, but with $n = 4$, so the probability of $x - y$ successes is:

$$P(X - Y = x - y) \ = \ \binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y}, \quad 0 \le x - y \le 4$$

Since the last four trials are independent of the first two, the probability of the joint event, $P(X = x \text{ and } Y = y)$ is the product:

$$\left[\binom{2}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{2-y}\right]\left[\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y}\right]$$

We may consider the six tosses as a binomial experiment with $n = 4$, so the probability of x successes is:

$$P(X = x) \ = \ \binom{6}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{6-x}, \quad 0 \le x \le 6$$

Now the conditional probability is by definition

$$p_{Y|x} \ = \ \frac{p_{XY}(x,y)}{p_X x}$$

which is

$$\frac{\left[\binom{2}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{2-y}\right]\left[\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y}\right]}{\binom{6}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{6-x}} \ = \ \frac{\binom{2}{y}\binom{4}{x-y}}{\binom{6}{x}}$$

which is a hypergeometric probability.

1) (tails) (Problem 3.11.3) An urn contains eight red chips, six white chips, and four blue chips. A sample of size 3 is drawn without replacement. Let X be the number of red chips, and Y the number of white chips. Find an expression for $P_{Y|x}(y)$.

Solution: Refering to the section on the hypergeometric distribution, it is suggested (problem 3.2.32) that the distribution generalizes to sampling from an urn with 3 kinds of objects, with the probability of the triple (k_1, k_2, k_3) given by

$$\frac{\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{n_3}{k_3}}{\binom{N}{n}}$$

so that, in this problem, the probability of obtaining y white and x red chips would be

$$P(X = x \,\text{and}\, Y = y) \;=\; \frac{\binom{6}{y}\binom{8}{x}\binom{4}{3-x-y}}{\binom{18}{3}}$$

To find the probability of choosing x red chips, consider the sample as a hypergeometric experiment with 8 red and 10 non-red chips, so

$$P(X = x) \;=\; \frac{\binom{8}{x}\binom{10}{3-x}}{\binom{18}{3}}$$

The conditional probability is the ratio of these two probabilities:

$$\frac{P(X = x \,\text{and}\, Y = y)}{P(X = x)} \;=\; \frac{\binom{6}{y}\binom{4}{3-x-y}}{\binom{10}{3-x}}$$

2)(heads) (Problem 3.11.11) A nonnegative random variable X is called *memoryless* if

$$P(X > s + t \,|\, X > t) \;=\; P(X > s) \quad \text{for all } s, t \geq 0$$

Show that a random variable with pdf

$$f_X(x) \;=\; \left(\frac{1}{\lambda}\right) e^{-x/\lambda}, \quad x > 0$$

is memoryless.

Solution: The conditional probability is

$$\frac{P(X > s + t \text{ and } X > t}{P(X > t)} \;=\; \frac{P(X > s + t)}{P(X > t)} \;=\; \frac{1 - F_X(s + t)}{1 - F_X(t)}$$

The cdf is

$$F_X(x) \;=\; \int_0^x \lambda e^{-\lambda x} dx \;=\; 1 - e^{-\lambda x}$$

so the conditional probability is

$$\frac{1 - F_X(s+t)}{1 - F_X(t)} \;=\; \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \;=\; e^{-\lambda s} \;=\; P(X > s)$$

2)(tails) (Problem 3.11.12) Given the joint pdf

$$F_{XY}(x, y) \;=\; 2e^{-x-y}, \quad 0 \,<\, x \,<\, y, \quad y > 0$$

find:

- (a) $P(Y < 1 \,|\, X < 1)$
- (b) $P(Y < 1 \,|\, X = 1)$
- (c) $f_{Y|x}(y)$
- (d) $E(Y \,|\, x)$

Solution: a) The marginal density of X is

$$f_X(x) \;=\; \int_x^\infty 2e^{-x-y} dy \;=\; 2e^{-2x}, \quad x > 0$$

Then

$$P(X < 1) \;=\; \int_0^1 2e^{-2x} dx \;=\; 1 - e^{-2} \;=\; 0.865$$

The probability of the joint event $Y < 1$ and $X < 1$ is

$$P(X < 1 \,\mathrm{and}\, Y < 1) \;=\; \int_0^1 \int_0^x 2e^{-x-y}\, dy\, dx \;=\; -2e^{-x} + e^{-2x}\Big|_0^1$$

$$= 0.4$$

The conditional probability is

$$\frac{0.4}{0.865} \;=\; 0.462$$

b) Since Y is always greater than X, this probability is zero.

c) The conditional density is

$$f_{Y|x} \;=\; \frac{f_{XY}(x, y)}{f_X(x)} \;=\; \frac{2e^{-x-y}}{2e^{-2x}} \;=\; e^x e^{-y} \; x < y$$

d) The conditional expectation is

$$\mathrm{E}(Y|x) \;=\; \int_x^\infty y \cdot e^x e^{-y} \;=\; (1+x)$$

3)(heads) (Problem 3.11.14) If

$$f_{XY}(x,y) \;=\; 2, \quad x \geq 0, \quad y \geq 0, \quad x+y \leq 1$$

show that the conditional pdf of Y given x is uniform.

Solution: The conditional density is given by

$$f_{Y|x}(y) \;=\; \frac{f_{XY}(x,y)}{f_X(x)}$$

We obtain the marginal of X by integrating $f_{XY}(x,y)$ over the range of values Y assumes, which is 0 to $1-x$,

$$f_X(x) \;=\; \int_0^{1-x} 2\,dy \;=\; 2y|_0^{1-x} \;=\; 2(1-x)$$

so

$$f_{Y|x}(y) \;=\; \frac{f_{XY}(x,y)}{f_X(x)} \;=\; \frac{2}{2(1-x)} \;=\; \frac{1}{1-x}$$

Since this is a constant for any fixed value of x, the conditional is a uniform density.

3)(tails) (Problem 3.11.15) Suppose that

$$f_{Y|x}(y) \;=\; \frac{2y+4x}{1+4x} \quad \text{and} \quad f_X(x) \;=\; \frac{1}{3}(1+4x), \quad 0 < x, y < 1$$

Find the marginal pdf of Y.

Solution: Use the definition of the conditional density

$$f_{Y|x}(y) \;=\; \frac{f_{XY}(x,y)}{f_X(x)}$$

to obtain an expression for the joint density:

$$f_{XY}(x,y) \;=\; f_{Y|x}(y) \cdot f_X(x) \;=\; \left(\frac{2y+4x}{1+4x}\right)\left(\frac{1}{3}(1+4x)\right) \;=\; \frac{2y+4x}{3}$$

Now find the marginal density of Y by integrating the joint density with respect to X:

$$f_Y(y) \;=\; \int_0^1 \frac{2y+4x}{3}dx \;=\; \frac{1}{3}\left(2xy+2x^2\right)\Big|_0^1$$

$$f_Y(y) \;=\; \frac{1}{3}\,(2y+2) \quad 0 \le y \le 1$$

4)(heads) (Problem 3.12.7) A random variable X has the Poisson distribution if

$$P(x=k) \;=\; p_X(k) \;=\; \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Find the moment generating function for a Poisson random variable.

Solution: From the definition, the moment generating function is

$$M_Y(t) \;=\; \mathrm{E}(e^{ty}) \;=\; \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!}; = \; e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= \; e^{-\lambda} e^{\lambda e^t} \;=\; e^{\lambda(e^t-1)}$$

4)(tails) (Problem 3.12.8) Let Y be a continuous random variable with

$$f_Y(y) \;=\; y e^{-y}, \quad 0 \le y$$

Show that

$$M_Y(t) \;=\; \frac{1}{(1-t)^2}$$

Solution: From the definition, $M_Y(t)$ is

$$M_Y(t) \;=\; \mathrm{E}(e^{ty}) \;=\; \int_0^{\infty} e^{ty} y e^{-y} \;=\; \int_0^{\infty} y e^{-y(1-t)} \;=\; \frac{1}{(1-t)^2}$$

5)(heads) (Problem 3.12.14) Find an expression for $\mathrm{E}(Y^k)$ if

$$M_Y(t) \;=\; (1 - t/\lambda)^{-r}$$

where λ is a positive real number and r is a positive integer.

Solution: The first derivative of $M_Y(t)$ with respect to t is

$$M_Y^{(1)}(0) = \frac{r}{\lambda}$$

The second derivative of $M_Y(t)$ with respect to t is

$$M_Y^{(2)}(0) = \frac{r(r+1)}{\lambda^2}$$

Continuing, the k^{th} derivative is The first derivative of $M_Y(t)$ with respect to t is

$$M_Y^{(k)}(t) = \frac{(r+k-1)!}{(r-1)!\lambda^k}$$

5)(tails) (Problem 3.12.16) Find the variance of Y if

$$M_Y(t) \;=\; \frac{e^{2t}}{1-t^2}$$

Solution: We will use the moment-generating function to find E(Y) and E(Y^2) then use the formula

$$\mathrm{Var}(Y) \;=\; \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2$$

Using the quotient rule, the first derivative of the moment-generating function is

$$M_Y^{(1)}(t) \;=\; \frac{(1-t^2)2e^{2t} - (-2t)e^{2t}}{(1-t^2)^2}$$

and so

$$\mathrm{E}(Y) \;=\; M_Y^{(1)}(0) \;=\; \frac{(1-0^2)2e^{0} - (-0)e^{0}}{(1-0^2)^2} \;=\; 2$$

A tedious but straightforward second differentation shows that

$$\mathrm{E}(Y^2) \;=\; M_Y^{(2)}(0) \;=\; 6$$

so

$$\mathrm{Var}(Y) \;=\; 6 - (2)^2 = 2$$

6)(heads) (Problem 3.12.19) Use theorems 3.12.2 and 3.12.3 to determine which of the following are true:

- a) The sum of two independent Poisson random variables has a Poisson distribution
- b) The sum of two independent exponential random variables has an exponential distribution
- c) The sum of two independent normal random variables has a normal distribution

Solution: a) The moment-generating function of the Poisson distribution is

$$M_Y(t) \;=\; \exp\left(-\lambda + \lambda e^t\right)$$

If X and Y are independent Poisson random variables with parameters μ and λ, respectively, then the moment-generating function of $X+Y$ is the product

$$M_{(X+Y)}(t) \;=\; M_X(t)\cdot M_Y(t) \;=\; \exp\left(-\lambda+\lambda e^t\right)\cdot\exp\left(-\mu+\mu e^t\right)$$
$$\;=\; \exp\left(-(\lambda+\mu)+(\lambda+\mu)e^t\right)$$

which is the moment-generating function of a Poisson random variable with parameter $\lambda+\mu$.

b) The moment-generating function of the exponential distribution is

$$M_Y(t) \;=\; \frac{\lambda}{\lambda-t}$$

If X and Y are independent exponential random variables with parameters μ and λ, respectively, then the moment-generating function of $X+Y$ is the product

$$M_{(X+Y)}(t) \;=\; M_X(t)\cdot M_Y(t) \;=\; \frac{\lambda}{\lambda-t}\cdot\frac{\mu}{\mu-t}$$
$$=\; \frac{\lambda\mu}{(\lambda-t)(\mu-t)}$$

which is not the moment-generating function of an exponential random variable, so the sum does not have an exponential distribution.

c) The moment-generating function of the normal distribution is

$$M_Y(t) \;=\; \exp(\mu t+\sigma^2t^2/2)$$

If X and Y are independent exponential random variables with parameters μ and λ, respectively, then the moment-generating function of $X+Y$ is the product

$$M_{(X+Y)}(t) \;=\; M_X(t)\cdot M_Y(t) \;=\; \exp(\mu_1t+\sigma_1^2t^2/2)\cdot\exp(\mu_2t+\sigma_2^2t^2/2)$$
$$M_{(X+Y)}(t) \;=\; \exp\left((\mu_1+\mu_2)t+(\sigma_1^2+\sigma_2^2)t^2/2\right)$$

which is the moment-generating function of a normal random variable with mean $\mu_1+\mu_2$ and variance $\sigma_1^2+\sigma_2^2$.

6)(tails) (Problem 3.12.21) Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a normal distribution with mean μ and standard deviation σ. Use moment-generating functions to determine the pdf of

$$\overline{Y} \;=\; \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Solution: The moment-generating function of a normal random variable with mean μ and standard deviation σ is

$$M_Y(t) \;=\; \exp(\mu t + \sigma^2 t^2/2)$$

Since n is a constant, by theorem 3.12.3a

$$M_Y(t) \text{ of } \frac{Y_i}{n} \quad \text{is} \quad M_Y(t/n) \;=\; \exp\left(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}\right)$$

The Y_i form a random sample, so they are independent, and by theorem 3.12.3b the moment-generating function of the sum of Y_i/n is the product of the individual moment-generating functions, so

$$M_{\overline{Y}}(t) \;=\; \prod_{i=1}^{n} \exp\left(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}\right) \;=\; \exp\left(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}\right)^n$$

$$=\; \exp\left(\frac{n\mu t}{n} + \frac{n\sigma^2 t^2}{2n^2}\right) \;=\; \exp\left(\mu t + \frac{\left(\frac{\sigma}{\sqrt{n}}\right)^2 t^2}{2}\right)$$

which we recognize as the moment-generating function of a normal random variable with mean μ and standard deviation $\sigma/\sqrt{n}$.