MA395 Takehome Quiz 5

Name:

1) (Problem 3.8.6) Let Y be a random variable with

$$f_Y(y) \;=\; 6y(1-y), \quad 0 \le y \le 1$$

Show that the pdf of $W = Y^2$ is

$$f_W(w) \;=\; 3(1-\sqrt{w})$$

Solution: One way to solve this problem is to observe that the CDF of W is

$$F_W(w) \;=\; P(W \le w) \;=\; P(Y^2 \le w) \;=\; P(Y \le \sqrt{w}) \;=\; F_Y(\sqrt{w})$$

The CDF of Y is

$$F_Y(y) \;=\; \int_0^y 6t(1-t)\,dt \;=\; 3y^2 - 2y^3$$

Then

$$F_W(w) \;=\; F_Y(\sqrt{w}) \;=\; 3\left(\sqrt{w}\right)^2 - 2\left(\sqrt{w}\right)^3$$

Taking the derivative with respec

Solution: Tt to w, we obtain

$$f_W(w) \;=\; 3(1-\sqrt{w})$$

A more general approach (not covered in the text) is the following:

Suppose:

- Y is a random variable with PDF $f_Y(y)$.
- h is a $1:1$ function differentiable on some open interval D that contains the support of Y.

Then

$$W = h(Y)$$

is a continuous random variable with density function

$$f_W(w) \;=\; f_Y\left(h^{-1}(w)\right) \cdot \left|\frac{d}{dw}h^{-1}(w)\right|$$

Now consider the previous example,

$$f_Y(y) \;=\; 6y(1-y), \quad 0 \le y \le 1$$

with

$$W \;=\; h(Y) \;=\; Y^2 \quad \text{and} \quad h^{-1}(w) \;=\; \sqrt{w}$$

Then

$$\frac{d}{dw}h^{-1}(w) \;=\; \frac{d}{dw}\sqrt{w} \;=\; \frac{1}{2\sqrt{w}}$$

then using the above theorem,

$$f_W(w) \;=\; 6\sqrt{w}\left(1-\sqrt{w}\right)\cdot\left|\frac{1}{2\sqrt{w}}\right| \;=\; 3\left(1-\sqrt{w}\right)$$

2) (Problem 3.8.2) Find the pdf of $X+Y$ if X and Y are independent random variables with

$$f_X(x) \;=\; xe^{-x}, \quad x \geq 0 \quad \text{and} \quad f_Y(y) = e^{-y}, \quad y \geq 0$$

Solution: The PDF of the sum is given by the convolution integral

$$f_{X+Y}(W) \;=\; \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)\,dx \;=\; \int_0^w \left(xe^{-x}\right)\left(e^{-(w-x)}\right)\,dx$$

$$= e^{-w}\int_0^w x\,dx \;=\; \frac{w^2}{2}e^{-w}, \quad w \geq 0$$

3) (Problem 3.9.2) Suppose

$$f_{XY}(x,y) \;=\; \lambda^2 \cdot e^{-\lambda(x+y)}$$

Find $E(X+Y)$.

Solution: Note that the joint PDF factors into the product of the marginals,

$$f_{XY}(x,y) \;=\; \lambda^2 \cdot e^{-\lambda(x+y)} \;=\; \left(\lambda e^{-x}\right)\left(\lambda e^{-y}\right)$$

so X and Y are independent exponential random variables, and we have a theorem that says for independent random variables X and Y,

$$\mathrm{E}(X+Y) \;=\; \mathrm{E}(X)+\mathrm{E}(Y) \;=\; \frac{1}{\lambda}+\frac{1}{\lambda} \;=\; \frac{2}{\lambda}$$

4) (Problem 3.8.7) Given that X and Y are independent random variables, find the pdf of XY for the following two sets of marginal pdfs:

(a) $f_X(x) = 1, \quad 0 \leq x \leq 1 \quad \text{and} \quad f_Y(y) = 1, \quad 0 \leq y \leq 1$

(b) $f_X(x) = 2x, \quad 0 \leq x \leq 1 \quad \text{and} \quad f_Y(y) = 2y, \quad 0 \leq y \leq 1$

Solution: (a) Let $V = XY$, then

$$f_V(v) \;=\; \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(v/x)\, f_Y(x)\, dx$$

Since $f_X(v/x) \neq 0$ when $0 \leq v/x \leq 1$, we only have to consider $v \leq x$.

Also, $f_Y(x) \neq 0$ implies $x \leq 1$, so that the integral becomes

$$\int_v^1 \frac{dx}{x} \;=\; \ln x|_v^1 \;=\; -\ln v, \quad 0 \leq v \leq 1$$

5) (Problem 3.9.13) Suppose

$$f_{XY}(x, y) \;=\; \lambda^2 \cdot e^{-\lambda(x+y)}$$

Find $\mathrm{Var}(X + Y)$.

Solution: As we saw earlier,

$$f_{XY}(x, y) \;=\; \lambda^2 \cdot e^{-(x+y)} \;=\; \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda y}\right)$$

so X and Y are independent exponential random variables, and we have a theorem that states that for independent random variables, the variance of their sum is the sum of their variances, so

$$\mathrm{Var}(X + Y) \;=\; \mathrm{Var}(X) + \mathrm{Var}(Y) \;=\; \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \;=\; \frac{2}{\lambda^2}$$