MA395 Takehome Quiz 4

Name:

1) Let X and Y be two continuous random variables defined over the unit square with joint pdf

$$f_{X,Y}(x,y) \quad = \quad c \cdot (x^2 + y^2)$$

a) Find the value of c.

b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

Solution: Integrating $f_{X,Y}(x,y)$ over the unit square should give 1, so

$$1 \;=\; c\int_0^1\int_0^1 (x^2+y^2)\,dydx \;=\; c\int_0^1\left[\int_0^1 (x^2+y^2)dy\right]dx$$

$$=\; c\int_0^1\left[\left(x^2y+\frac{y^3}{3}\right)\Big|_0^3\right]dx \;=\; c\int_0^1\left[x^2+\frac{1}{3}\right]dx \;=\; c\left(\frac{x^3}{3}+\frac{x}{3}\right)\Big|_0^1 \;=\; \frac{2}{3}c$$

so $c = 3/2$. The marginals are:

$$f_Y(y) \;=\; \frac{3}{2}\int_0^1 (x^2+y^2)\,dx \;=\; \frac{3}{2}\left(\frac{x^3}{3}+xy^2\right)\Big|_0^1 \;=\; \frac{1}{2}\left(1+3y^2\right)$$

$$f_X(x) \;=\; \frac{3}{2}\int_0^1 (x^2+y^2)\,dy \;=\; \frac{3}{2}\left(\frac{y^3}{3}+x^2y\right)\Big|_0^1 \;=\; \frac{1}{2}\left(1+3x^2\right)$$

2) Suppose that random variables X and Y vary in accordance with the joint pdf

$$f_{XY}(x,y) \quad = \quad c\cdot(x+y), \quad 0 \;<\; x \;<\; y\;1$$

a) Find c.

b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

Solution: The double integral of $f_{XY}(x,y)$ over its support must be 1, so

$$1 \;=\; \int_0^1\int_0^y c(x+y)\,dxdy \;=\; c\int_0^1\left[\left(\frac{x^2}{2}+xy\right)\Big|_0^y\right]dy$$

$$=\; c\int_0^1 \frac{3y^2}{2}\,dy \;=\; c\,\frac{y^3}{2}\Big|_0^1 \;=\; \frac{c}{2}$$

so $c = 2$. To find the marginal of Y, $f_Y(y)$, we integrate $f_{XY}(x, y)$ with respect to X over the support. In this case, there is no support if $Y < X$, so the appropriate integral is:

$$f_Y(y) \;=\; 2\int_y^1 (x + y)\,dx \;=\; \left(2xy + x^2\right)\Big|_y^1 \;=\; 1 + 2y - 3y^2$$

To find the marginal of X, $f_X(x)$, we integrate $f_{XY}(x, y)$ with respect to Y over the support. In this case, there is no support if $Y < X$, so the appropriate integral is:

$$f_X(x) \;=\; 2\int_0^x (x + y)\,dy \;=\; \left(2xy + y^2\right)\Big|_0^x \;=\; 3x^2$$

3) Consider the experiment of tossing a fair coin three times. Let X denote the number of heads obtained on the last flip, and let Y denote the total number of heads in three flips. Find $f_{X,Y}(x, y)$.

Solution: The set of possible outcomes is given by the following table:

Outcome	X	Y
HHH	1	3
HHT	0	2
HTH	1	2
HTT	0	1
THH	1	2
THT	0	1
TTH	1	1
TTT	0	0

Each event in the table has probability 1/8. Next construct a table of probabilities for the values of (X, Y) that accually occur, which represents $f_{XY}(x, y)$.

(x, y)	$f_{XY}(x, y)$
(0,0)	1/8
(0,1)	2/8
(0,2)	1/8
(1,1)	1/8
(1,2)	2/8
(1,3)	1/8

4) (Problem 3.7.12) A point is chosen at random from the interior of the circle whose equation is

$$x^2 + y^2 \ = \ 4$$

Let the random variables X and Y be the x-coordinate and y-coordinate, respectively, of the point chosen.

a) Find $f_{X,Y}(x, y)$.

b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

Solution: The distribution is a joint uniform density, so it will be a constant function. The (constant) value of $f_{XY}(x, y)$ will be the height of a cylinder with unit volume whose base is the circle of radius 2 centered at the origin.

Since the volume of such a cylinder is $4\pi h$, $h = f_{XY}(x, y)$ must be $1/4\pi$.

One way to find the marginal of X is to find the marginal CDF $F_X(x)$ and take its derivative. By definition, $F_X(x)$ is the probability that $X \leq x$, which is $1/4\pi$ times the part of the area of the circle that lies to the left of x, or

$$\frac{2}{4\pi} \int_{-2}^{x} \sqrt{4 - t^2} \, dt$$

By the Fundamental Theorem of Calculus, the derivative of this integral with respect to x is just the integrand evaluated at $t = x$, so

$$f_X(x) \ = \ \frac{1}{2\pi} \sqrt{4 - x^2}$$

By symmetry, the marginal of Y is

$$f_Y(y) \ = \ \frac{1}{2\pi} \sqrt{4 - y^2}$$

5) Suppose X and Y are random variables with joint pdf $f_{X,Y}(x, y) = x + y$ for X and Y each defined over the unit interval. Find

$$P(X < 2Y)$$

(i.e., find the probability of the event that X is smaller than $2Y$)

Solution: The support in this case is the unit square, and the event corresponds to the portion of the unit square that lies above the line

$y = x/2$. The probability of the event $X < 2Y$ corresponds to the area of this region, or

$$\int_0^1 \int_{x/2}^1 (x+y)\, dy dx \;=\; \int_0^1 \left[\int_{x/2}^1 (x+y)\, dy \right] dx$$

$$\int_0^1 \left[\left(xy + \frac{y^2}{2} \right) \Big|_{x/2}^1 \right] dx \;=\; \int_0^1 \left(x + \frac{1}{2} - \frac{x^2}{2} - \frac{x^2}{8} \right) dx$$

$$= \left(x + \frac{1}{2} - \frac{x^2}{2} - \frac{x^2}{8} \right) \Big|_0^1 \;=\; \frac{1}{2} + \frac{1}{2} - \frac{1}{6} - \frac{1}{24} \;=\; \frac{19}{24}$$