MA395 Takehome Quiz 3

Name:

1) A fair coin is tossed until the first tail appears. The payoffs are:

$$\text{payoff} \;=\; \begin{cases} \$2^k & 1 \le k \le 9 \\ \$1,000 & k \ge 10 \end{cases}$$

How much should you have to pay to play the game to make it "fair" (i.e., to make your expected winnings equal to the cost to play the game)?

Solution: The "fair" cost to play is the expected value of the payoff,

$$\text{E}(X) \;=\; \sum_{k=1}^{9} 2^k \left(\frac{1}{2^k}\right) \;+\; \sum_{k=10}^{\infty} 1000 \left(\frac{1}{2^k}\right) \;=\; 9 + \frac{1000}{2^{10}} \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right)$$

$$= 9 + \frac{1000 \cdot 2}{1024} \;=\; \$10.95$$

2) Show that if the random variable Y has the exponential distribution

$$f_Y(y) = \lambda e^{-\lambda x} \qquad y > 0$$

then

$$\text{E}(Y) = \frac{1}{\lambda} \quad \text{and} \quad \text{Var}(Y) = \frac{1}{\lambda^2}$$

Solution: From the definition,

$$\text{E}(Y) \;=\; \int_0^{\infty} \lambda y \cdot e^{-\lambda y}\, dy \;=\; -\left(\frac{(1+\lambda y)e^{-\lambda y}}{\lambda}\right)\Bigg|_0^{\infty} \;=\; \frac{1}{\lambda}$$

$$\text{E}(Y^2) \;=\; \int_0^{\infty} \lambda y^2 \cdot e^{-\lambda y}\, dy \;=\; -\left(\frac{(2+2\lambda y+\lambda^2 y^2)e^{-\lambda y}}{\lambda^2}\right)\Bigg|_0^{\infty} \;=\; \frac{2}{\lambda^2}$$

$$\text{Var}(Y) \;=\; \text{E}(Y^2) - [\text{E}(Y)]^2 \;=\; \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \;=\; \frac{1}{\lambda^2}$$

3) If $\text{E}(W) = \mu$ and $\text{Var}(X) = \sigma^2$ show that

$$\text{E}\left(\frac{W-\mu}{\sigma}\right) = 0 \quad \text{and} \quad \text{Var}\left(\frac{W-\mu}{\sigma}\right) = 1$$

Solution: Using Theorem 3.6.2,

$$\mathrm{E}\left(\frac{W-\mu}{\sigma}\right) = \left(\frac{1}{\sigma^2}\right)[\mathrm{E}(W-\mu)] = \left(\frac{1}{\sigma^2}\right)[\mathrm{E}(W)-\mu] = 0$$

$$\mathrm{Var}\left(\frac{W-\mu}{\sigma}\right) = \left(\frac{1}{\sigma^2}\right)\mathrm{Var}(W) = 1$$

4) Let Y be a uniform random variable defined over the interval $(0, 2)$. Find an expression for the r^{th} moment of Y about the origin.

Solution: From the definition,

$$\mathrm{E}\left(Y^r\right) = \int_0^2 \frac{y^r dy}{2} = \left(\frac{1}{2}\right) \left.\frac{y^{r+1}}{r+1}\right|_0^2 = \frac{2^r}{r+1}$$

5) Suppose a random variable Y has odf

$$f_Y(y) = c \cdot y^{-6}, \quad y > 1$$

a) Find c.

b) What is the highest moment of Y that exists?

Solution: First find c by setting the integral of $f_Y(y)$ over its support to 1:

$$1 = \int_1^\infty cy^{-6}dy = c \cdot \left.\frac{y^{-5}}{-5}\right|_1^\infty = \frac{c}{5}$$

so $c = 5$. Now from the definition,

$$\mathrm{E}\left(Y^r\right) = 5\int_1 \infty y^r y^{-6}dy = 5 \cdot \left.\frac{y^{r-5}}{r-5}\right|_0^\infty$$

The rightmost expression is only finite if $r < 5$, so the highest moment that exists is the fourth.