MA395 Takehome Quiz 2

Name:

1) Let n be any positive integer. Show that, for a given n, the function

$$f_Y(y) = (n+2)(n+1)y^n(1-y), \quad 0 \le y \le 1$$

is a pdf.

Solution: The function is nonnegative on $[0,1]$. Now, integrating $f_Y(y)$ over its support,

$$\int_0^1 f_Y(y) \;=\; (n{+}1)(n{+}2)\int_0^1 \left(y^n - y^{n+1}\right) dy \;=\; (n{+}2)(n{+}1)\left(\frac{y^{n+1}}{n+1} - \frac{y^{n+2}}{n+2}\right)\Bigg|_0^1$$

$$= \left[(n+2)y^{n+1} - (n+1)y^{n+2}\right]\Big|_0^1 \;=\; 1$$

2) Suppose Y is an exponential random variable, so that

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \ge 0$$

Find the cumulative distribution function $F_Y(y)$.

Solution:

$$F_Y(y) \;=\; P(Y \le y) \;=\; \int_0^y \lambda \cdot e^{-\lambda t} \;=\; -e^{-\lambda t}\Big|_0^y \;=\; 1 - e^{-\lambda y}$$

3) Suppose Y is a random variable representing the time to failure for a machine. The **hazard rate** is defined to be the probability that an item fails at time y, *given that it has survived until time* y. In terms of the pdf and cdf of Y, the hazard rate is:

$$h(y) = \frac{f_Y(y)}{1 - F_Y(y)}$$

Find $h(y)$ if Y has an exponential distribution.

Solution: In this case the hazard ratio is:

$$h(y) \;=\; \frac{\lambda e^{-\lambda y}}{1 - (1 - e^{-\lambda y})} \;=\; \frac{\lambda e^{-\lambda y}}{e^{-\lambda y}} \;=\; \lambda$$

4) Find a constant α so that, for some given value $k > 1$,

$$f_Y(y) = \frac{\alpha}{y^2}, \quad y \ge k$$

is a pdf.

Solution: For $k > 1$, $f_Y(y)$ is nonnegative, so we only have to choose α to make the integral of $f_Y(y)$ over its support equal to 1,

$$\int_k^\infty \frac{\alpha}{y^2}\, dy \;=\; -\frac{\alpha}{k}\Big|_k^\infty \;=\; \frac{\alpha}{k}$$

so, choose $\alpha = k$.

5) Suppose

$$F_Y(y) = \frac{1}{12}(y^2 + y^3), \quad 0 \le y \le 2$$

find $f_Y(y)$.

Solution:

$$f_Y(y) \;=\; \frac{d}{dy}\left[\frac{y^2 + y^3}{12}\right] \;=\; \frac{y}{6} + \frac{y^2}{4}, \quad 0 \le y \le 2$$