Contents

VECTOR SPACES

1. Norms

From the previous set of notes, recall the definition of a norm:

Definition 1 (norm). A nonnegative real-valued function $\| \ \| : V \to \mathbb{R}$ is called a **norm** if:

- $\|v\| \geq 0$ and $\|v\| = 0 \ \Leftrightarrow \ v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\| \quad$ (triangle inequality)
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F, \ v \in V$

A linear space V together with a norm $\| \cdot \|$, denoted by the pair $(V, \| \cdot \|)$, is called a **normed linear space**

1.1. Examples.

A Euclidean space $\mathbb{R}^{\ltimes}$ can be made into a normed space in a number of ways.

The "usual norm" in Euclidean space is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Among the alternatives are:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|x\|_\infty = \max(x_1, \ldots, x_n)$$

In fact a family of norms $\| \cdot \|_p$ for $1 \leq p < \infty$ is defined by

$$\|x\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

and $\|x_p\|$ tends to $\|x\|_\infty$ as $p \to \infty$

Each of these norms can be combinded with the set of sequences $x = \{x_n\}$ for which the the expression for $\|x\|$ is finite to produce a normed space.

The space of bounded sequences becomes a normed space with norm

$$\|x\|_\infty = \sup_{1 \leq n < \infty} |x_n|$$

If S is a nonempty set and $B(S)$ is the space of bounded real (or complex) valued functions on S,

$$B(S) = \{f | f : S \to \mathbb{R}\} \quad \text{or} \quad \{f | f : S \to \mathbb{C}\}$$

then the following defines a norm on $B(S)$:

$$\|f\|_S = \sup_{s \in S} |f(s)|$$

This is known as the *sup norm* over S.

The space of continuous functions on $[0, 1]$ can be made into a normed space with the sup norm, or others such as

$$\|f\|_1 = \int_0^1 |f(s)| ds$$

or

$$\|f\|_p = \left(\int_0^1 |f(s)|^p ds \right)^{\frac{1}{p}}$$

2. Norms and Metrics

Definition 2 (metric). A **metric** on a set X is a real-valued function $\rho : X \times X \to \mathbb{R}$ with the following properties:

$$\begin{array}{ll} \rho(x,y) \geq 0 & \forall x, y \in X \\ \rho(x,y) = 0 \Leftrightarrow x = y & \forall x, y \in X \\ \rho(x,y) = \rho(y,x) & \forall x, y \in X \\ \rho(x,z) \leq \rho(x,y) + \rho(y,z) & \forall x, y, z \in X \end{array}$$

Theorem 1. If $(X, \| \cdot \|)$ is a normed linear space, then

$$\rho(x,y) = \|x - y\| \quad \forall x, y \in X$$

is a metric on X.

By the previous theorem, a norm $\|\cdot\|$ on a linear space induces a metric ρ, so a normed linear space $(X, \|\cdot\|)$ can be thought of as a metric space (X, ρ).

If we speak of the metric properties of a normed linear space, it is with respect to the metric induced by the norm.

If this metric space has the property that every Cauchy sequence converges, it is said to be *complete*.

Definition 3 (Banach space). A normed linear space that is complete in the metric induced by the norm is called a **Banach space**

In a normed linear space, the norm induces a metric. The metric in turn induces a topoloy known as the*metric topology*. A set of the form

$$B(x, \epsilon) = \{y : \rho(x, y) < \epsilon\}$$

known as an ϵ-neighborhood or ϵ-ball centered at x. In the topology induced by the metric ρ, a set U is open if and only if for every $x \in U$, there exists an ϵ-neighborhood $B(x, \epsilon)$ centered at x that is entirely contained in U:

$$\exists B(x, \epsilon) \subset U$$

3. Inner Product Spaces

Another feature that a linear space may or may not have is a binary operation $\cdot : V \times V \to F$ called a *scalar product* or *inner product* or sometimes *dot product*.

Definition 4 (inner product). An **inner product** or **scalar product** on a vector space V over $\mathbb{R}$ is a map $\cdot : V \times V \to \mathbb{R}$ with the following properties:

$$\begin{array}{ll} x \cdot y = y \cdot x & \forall x, y \in V \\ \alpha x \cdot y = \alpha(x \cdot y) & \forall \alpha \in \mathbb{R},\ x, y \in V \\ (x + y) \cdot z = x \cdot z + y \cdot z & \forall x, y, z \in V \\ (x, x) \geq 0 & \forall x \in V \text{ with equality when } x = 0 \end{array}$$

A linear space with an inner product defined is called an **inner product space**.

3.1. **Examples.** In Euclidean space $\mathbb{R}^n$,

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

and also

$$x \cdot x = x_1^2 + \cdots + x_n^2 \;=\; \|x\|^2$$

where $\|\cdot\|$ is the Euclidean norm:

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}, \quad x \in \mathbb{R}^n$$

In the space of continuous functions on $[0,1]$, $C[0,1]$,

$$f \cdot g = \int_0^1 fg \; dx$$