Contents

INNER PRODUCT SPACES

1. Inner Products

Restricting our attention to linear spaces where the field F is either $\mathbb{R}$ or $\mathbb{C}$, we will consider vector spaces in which it makes sense to speak of the *length* of a vector, as well as the *angle* between vectors and orthogonality.

The mechanism for this is a binary operation or function on vectors that produces a scalar,

$$(\cdot) \ : \ V \times V \to F$$

which is known as an *inner product*.

Definition 1 (inner product). Let the field F be either $\mathbb{R}$ or $\mathbb{C}$ and a set V of vectors which together with F form a vector space. An **inner product** on V is a map

$$\cdot : V \times V \to \mathbb{F}$$

with the following properties:

$$\begin{array}{ll} (u+v)\cdot w = u\cdot w \ + \ v\cdot w & \forall u, v, w \in V \\ (\alpha u)\cdot v = \alpha(u\cdot v) & \forall \alpha \in F, \ u, v \in V \\ u\cdot v = (\overline{v\cdot u}) & \forall u, v \in V \\ u\cdot u \geq 0 & \forall u \in V \text{ with equality when } u = \vec{0} \end{array}$$

A linear space which has an inner product defined is called an **inner product space**.

A finite-dimensional inner product space over $\mathbb{R}$ is usually called a **Euclidean space**.

An inner product space over $\mathbb{C}$ is usually called a **unitary space**.

Here the bar $\overline{\alpha}$ refers to the complex conjugate, that is, if $\alpha = a + bi$ then $\overline{\alpha} = a - bi$. If $F = \mathbb{R}$, then $b = 0$ for all elements of F so each scalar α is its own complex conjugate and the third property reduces to

$$u \cdot v = v \cdot u$$

So in a vector space over the reals, inner products are commutative, but this is not the case when the underlying field is $\mathbb{C}$.

It is necessary to define the third property differently for a complex vector space because if the inner product was commutative in a complex vector space, the fourth property, which requires

$$u \cdot u \geq 0$$

would be contradicted by

$$iu \cdot iu \ = \ i(u \cdot iu) \ = \ i(iu \cdot u) \ = \ i^2(u \cdot u) \ = \ -u \cdot u$$

Instead, if $(u \cdot v) = (\overline{v \cdot u})$, the result is

$$iu \cdot iu \ = \ i(u \cdot iu) \ = \ i(-iu \cdot u) \ = \ -i^2(u \cdot u) \ = \ u \cdot u$$

1.1. **Inner Product Spaces.** A linear space with an inner product defined is called an **inner product space**.

1.2. **Examples.**

Example 1. If $V = \mathbb{C}^n$, an inner product can be defined by

$$u \cdot v \ = \ u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}$$

Example 2. If $V = \mathbb{R}^n$, an inner product can be defined by

$$u \cdot v \ = \ u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example 3. If $V = F^{1\times n}$ represents the set of $1 \times n$ matrices (or row vectors) whose elements belong to F, and Q is an $n \times n$ invertible matrix over F, then for $u, v \in V$, the following defines an inner product:

$$u \cdot v \ = uQQ^* v^*$$

Here A^* represents the *conjugate transpose* of A, that is, if

$$A \ = \ a_{ij} \quad \text{then} \quad A^* \ = \ \overline{a}_{ji}$$

If Q is the identity matrix, this inner product is identical to the first example when $F = \mathbb{C}$ and the previous example when $F = \mathbb{R}$.

Example 4. If V is the space of all continuous complex-valued functions on $[0, 1]$, an inner product can be defined by

$$f \cdot g \;=\; \int_0^1 f(t)\overline{g(t)}dt$$

Example 5. If V is the space of all continuous real-valued functions on $[0, 1]$, an inner product can be defined by

$$f \cdot g \;=\; \int_0^1 f(t)\, g(t)\, dt$$

1.3. **Inner products and norms.** If V is an inner product space, the positive square root of the inner product of a vector with itself

$$\sqrt{u \cdot u} \;=\; \|u\|$$

is called the **norm** of u with respect to the inner product.

Example 6. If $V = \mathbb{R}^n$ equipped with the inner product defined by

$$u \cdot v \;=\; u_1 v_1 + \cdots + u_n v_n$$

then

$$\sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \|u\|$$

Example 7. If $V = \mathbb{C}^n$ equipped with the inner product defined by

$$u \cdot v \;=\; u_1 \overline{v_1} + \cdots + u_n \overline{v_n}$$

then

$$\sqrt{u \cdot u} = \sqrt{u_1 \overline{u_1} + \cdots + u_n \overline{u_n}} \;=\; \sqrt{|u_1|^2 + \cdots + |u_n|^2} = \|u\|$$

(where $|z|^2 = z\overline{z}$ for all $z \in \mathbb{C}$)

If a linear space over $\mathbb{R}$ or $\mathbb{C}$ has an inner product defined, the inner product induces a norm.

Theorem 1. If V is an inner product space, then for any vectors u and v and any scalar α,

1. $\|\alpha v\| = |\alpha|\|v\|$
2. $\|v\| \geq 0 \quad \forall v \in V, \quad \|v\| = 0 \Leftrightarrow v = \vec{0}$
3. $|u \cdot v| \leq \|u\|\|v\|$
4. $\|u + v\| \leq \|u\| + \|v\|$