Contents

THE TOPOLOGY OF $\mathbb{R}^n$

Recall the following definition of a topology:

Definition 1 (topology). A **topology** is a set X and a collection $\mathcal{J}$ of subsets of X having the following properties:

- $\emptyset$ and X are in $\mathcal{J}$
- The union of any subcollection of elements of $\mathcal{J}$ belongs to $\mathbb{J}$
- The intersection of any *finite* subcollection of $\mathcal{J}$ belongs to $\mathcal{J}$

The elements of the collection $\mathcal{J}$ are called **open sets**.

Recall that we have defined an inner product on $\mathbb{R}^n$, and as a result

$$\|x\| = \sqrt{x \cdot x}$$

is a norm on $\mathbb{R}^n$, and $dist(x, y) = \|x - y\|$ is a metric or distance measure.

We will now proceed to construct a topology on the set $\mathbb{R}^n$ by defining which subsets of $\mathbb{R}^n$ are open. The development follows a standard technique that can be used to define a topology in any metric space, known as the *metric topology*. Many other topologies are possible on $\mathbb{R}^n$, but only the one we are about to define arises directly from the metric.

Definition 2 (open ball). Let a be an arbitrary element of $\mathbb{R}^n$ and r a positive real number. The **open ball** centered at a with radius r is the set of points

$$B_r(a) = \{x \in \mathbb{R}^n \ : \ \|x - a\| < r\}$$

Now we can define an open set in $\mathbb{R}^n$ as follows:

Definition 3 (open set in $\mathbb{R}^n$). A subset O of $\mathbb{R}^n$ is said to be **open** if and only if for every $x \in O$ there is an $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq V$$

There are a number of ways to define a closed set. Last semester we defined a closed set in $\mathbb{R}$ as a set that contained its limit points, and then proved a theorem which stated that a subset of $\mathbb{R}$ is closed if and only if its compliment in $\mathbb{R}$ is open.

In general, an if and only if theorem can serve as a definition. This time we will start by *defining* a closed set to be the compliment of an open set:

Definition 4 (closed set in $\mathbb{R}^n$). A subset F of $\mathbb{R}^n$ is said to be **closed** if and only if $\mathbb{R}^n \setminus F$, the compliment of F in $\mathbb{R}^n$, is open.

Theorem 1. An open ball in $\mathbb{R}^n$ is an open set.

Proof. Let x be an arbitrary element of $B_r(a)$. Then by definition

$$\|x - a\| < r \quad \text{so} \quad 0 < r - \|x - a\|$$

Let $\epsilon = r - \|x - a\|$. Then if $y \in B_\epsilon(x)$,

$$\|y - x\| < \epsilon$$

so

$$\|y - a\| \leq \|y - x\| + \|x - a\| < \epsilon + \|x - a\| = r$$

which means $y \in B_r(a)$. Since y was an arbitrary choice, every $y \in B_\epsilon(x)$ is also in $B_r(a)$, so

$$B_\epsilon x \subseteq B_r(a)$$

□

Theorem 2. The empty set $\emptyset$ and $\mathbb{R}^n$ are both open and closed.

Proof. $\mathbb{R}^n$ is open, because for every $x \in \mathbb{R}^n$ and $\epsilon > 0$, $B_\epsilon(X) \subseteq \mathbb{R}^n$. This means $\emptyset$ is closed. However, $\emptyset$ satisfies the condition that it an open ball around every element of $\emptyset$ *vacuously.* So $\emptyset$ is open, and therefore $\mathbb{R}^n$ is closed. □

In topology, open and closed are not mutually exclusive. A set can be open, closed, both open and closed, or neither open nor closed.

Definition 5 (singleton)*.* A **singleton** is a set consisting of a single element.

Theorem 3*.* Singletons are closed in $\mathbb{R}^{\ltimes}$

Proof. Let x be a single element of $\mathbb{R}^n$ and let y be any other element. Then if $F = \{x\}$, y belongs to the compliment of F. Let $\epsilon = \|x - y\|$. Then since $x \neq y$, $\epsilon > 0$, and

$$B_\epsilon(y) \subseteq F^c$$

Since y was an arbitrary choice, there is an open ball around every element of F^c that is contained in F^c, so F^c is open, and therefore F is closed. □