Contents

EUCLIDEAN SPACES

1. Linear Functions on $\mathbb{R}^n$

Now consider a class of functions known as *linear* functions. The defining characteristic of these functions can be seen in the following special case: Let $T : \mathbb{R} \to \mathbb{R}$ have the folowing properties:

$$T(x+y) = T(x) + T(y) \quad \text{and} \quad T(\alpha x) = \alpha T(x) \quad \forall \alpha, x, y \in \mathbb{R}$$

The graph of a function of this type is always a straight line passing through the origin, which we can establish by the following result:

Theorem 1. Let $T : \mathbb{R} \to \mathbb{R}$ have the following properties:

$$T(x+y) = T(x) + T(y) \quad \text{and} \quad T(\alpha x) = \alpha T(x) \quad \forall \alpha, x, y \in \mathbb{R}$$

Then there exists an $s \in \mathbb{R}$ such that

$$T(x) = sx \quad \forall x \in \mathbb{R}$$

Proof. Let $s = T(1)$. Then for an arbitrary $x \in \mathbb{R}$,

$$T(x) = T(x \cdot 1)$$

Now both x and 1 are scalars, so we can treat x as a constant multiplier α and write

$$T(x \cdot 1) = xT(1) = xs$$

□

Now we consider the generalization of this class of functions to Euclidean space $\mathbb{R}^n$:

Definition 1 (linear function). A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if and only if it satisfies

$$T(x+y) = T(x) + T(y) \quad \text{and} \quad T(\alpha x) = \alpha T(x) \quad \forall x, y \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R}$$

Functions of this type are also called *linear transformations*. Collectively the set of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are denoted by

$$\mathcal{L}(\mathbb{R}^n : \mathbb{R}^m)$$

It can be shown that to every element of this set there exists a corresponding $m \times n$ matrix B with the property that

$$T(x) = B[x]^T$$

where $[x]^T$ represents the vector $x = (x_1, x_2, \ldots, x_n)$ written as an $n \times 1$ column vector.

It should be clear from the properties of matrix multiplication that $T(x) = Bx^T$ is indeed a linear function:

$$T(x+y) = B(x+y)^T = Bx^T + By^T$$

and

$$T(\alpha x) = B(\alpha x^T) = (\alpha B)x^T = \alpha(Bx^T) = \alpha T(x)$$

where αB is the matrix B with each element multiplied by α.

As we noted earlier, the set of $m \times n$ real matrices for a group with respect to elementwise addition, and because of this $\mathcal{L}(\mathbb{R}^n : \mathbb{R}^m)$ is a linear space with the corresponding addition operation.

Furthermore, we can define a norm on $\mathcal{L}(\mathbb{R}^n : \mathbb{R}^m)$, called the *operator norm* as follows:

Definition 2 (operator norm). The **operator norm** of an element T of $\mathcal{L}(\mathbb{R}^n : \mathbb{R}^m)$ is the extended real number

$$\|T\| = \sup_{\|x\| \neq 0} \frac{\|T(x)\|}{\|x\|}$$

As it turns out, the operator norm is finite:

Theorem 2. Let T be an element of $\mathcal{L}(\mathbb{R}^n : \mathbb{R}^m)$. Then $\|T\|$ is finite and satisfies

$$\|Tx\| \leq \|T\| \|x\| \quad \forall x \in \mathbb{R}^n$$

Proof. As noted for each T there is an $m \times n$ real matrix B such that $Tx = Bx^T$. If we think of the m rows of matrix B as row vectors $b_1, b_2, \ldots, b_m$, then we can write Tx as an m-dimensional vector

$$Tx = Bx^t = (b_1 \cdot x, b_2 \cdot x, \ldots, b_m \cdot x)$$

and so by definition

$$\|Tx\|^2 \;=\; Tx \cdot Tx \;=\; (b_1 \cdot x, \ldots, b_m \cdot x) \cdot (b_1 \cdot x, \ldots, b_m \cdot x)$$

so

$$\|Tx\|^2 \;=\; (b_1 \cdot x)^2 + \cdots + (b_m \cdot x)^2$$

Applying the Cauchy-Schwartz inequality we can write

$$\|Tx\|^2 \;=\; (b_1 \cdot x)^2 + \cdots + (b_m \cdot x)^2 \leq (\|b_1\|\|x\|)^2 + \cdots + (\|b_m\|\|x\|)^2)$$

$$\leq \max\{(\|b_1\|^2, \ldots, \|b_m\|^2\} \cdot m\|x\|^2 \;=\; C\|x\|^2$$

So $\|Tx\|^2$ is bounded, and therefore so is $\|Tx\|$. □