Contents

EUCLIDEAN SPACES

1. $\mathbb{R}^n$ as a Linear Space

Now we consider the linear space $\mathbb{R}^n$ which consists of:

- The field of real numbers $\mathbb{R}$ (scalars)
- The commutative group (with elementwise addition) of n-tuples of real numbers $\mathbb{R}^n$ (vectors)
- The scalar multiplication operation : $\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ (elementwise multiplication)

Note that in Definition 8.1, the author defines the sum, difference, and (scalar) product.

The *sum* is the binary operation

$$+ : \mathbb{R}^n \to \mathbb{R}^n$$

defined for $x = (x_1, \ldots, x_n)$ and $y = y_1, \ldots, y_n)$ as:

$$x + y \;=\; (x_1 + y_1, \ldots, x_n + y_n)$$

The author defines the *difference* as the binary operation

$$- : \mathbb{R}^n \to \mathbb{R}^n$$

defined for $x = (x_1, \ldots, x_n)$ and $y = y_1, \ldots, y_n)$ as:

$$x - y \;=\; (x_1 - y_1, \ldots, x_n - y_n)$$

Theorem 1. The set of vectors $\mathbb{R}^n$ form a commutative group with respect to the operation $+$, the zero element being the vector of n zeros: $(0, \ldots, 0)$.

Proof. First note that since the scalar field $\mathbb{R}$ is closed under addition,

$$x + y \;=\; (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n \quad \forall x, y \in \mathbb{R}^n$$

so $\mathbb{R}^n$ is closed under the operation $+$.

The addition operation is associative, which follows from the fact that addition in the underlying scalar field $\mathbb{R}$ is associative:

$$(x+y)+z \;=\; (x_1+y_1, \ldots, x_n+y_n)+(z_1, \ldots, z_n) \;=\; (x_1+y_1+z_1, \ldots, x_n+y_n+z_n)$$

$$= \; (x_1 + (y_1 + z_1), \ldots, x_n + (y_n + z_n)) = x + (y + z) \quad \forall x, y, z \in \mathbb{R}^n$$

The existence of a zero element in $\mathbb{R}^n$ also follows because the underlying field $\mathbb{R}$ has a zero element, so if we let

$$\vec{0} = (0, \ldots, 0)$$

then

$$x + \vec{0} \;=\; (x_1 + 0, \ldots, x_n + 0) \;=\; (0 + x_1, \ldots, 0 + x_n) \;==\; (x_1, \ldots, x_n)$$

so

$$x + \vec{0} = \vec{0} + x = x \quad \forall x \in \mathbb{R}^n$$

Next, the existence of a additive inverses in $\mathbb{R}$ means there are additive inverses in $\mathbb{R}^n$. Let

$$x = (x_1, \ldots, x_n) \quad \text{and} \quad -x = (-x_1, \ldots, -x_n)$$

where $-x_i = (-1) * x_i$ is the additive inverse of x_i in the scalar field, that is, the unique element of $\mathbb{R}$ such that $x_i + (-x_i) = 0$. Then

$$x+(-x) \;=\; (x_1+(-x_1), \ldots, x_n+(-x_n)) \;=\; (0, \ldots, 0) \;=\; \vec{0} \quad \forall x \in \mathbb{R}^n$$

Finally, note that because addition in the scalar field $\mathbb{R}$ is commutative, so is addition in $\mathbb{R}^n$:

$$x+y \;=\; (x_1+y_1, \ldots, x_n+y_n) \;=\; (y_1+x_1, \ldots, y_n+x_n) = y+x \quad \forall x, y \in \mathbb{R}^n$$

□

Now we can define the difference $x - y$ in terms of the addition operation on the group of vectors $\mathbb{R}^n$ as the vector sum of x and the additive inverse of y,

$$x - y \;=\; x + (-y) \quad \forall x, y \in \mathbb{R}^n$$

Multiplication of a vector by a scalar is defined as follows. Let

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

be an arbitrary vector and $\alpha \in \mathbb{R}$ an arbitrary scalar. Then

$$\alpha x = (\alpha x_1, \ldots, \alpha x_n)$$

where the multiplications within the parentheses refer to the product in the underlying field $\mathbb{R}$, while the multiplication on the left hand side is the "multiplication of a vector by a scalar" operation that the definition of a linear space requires.

2. Inner Products and Norms on $\mathbb{R}^n$

Recall that an **inner product** on a vector space V over the field $\mathbb{R}$ or $\mathbb{C}$ is a map

$$\cdot : V \times V \to \mathbb{F}$$

with the following properties:

$$\begin{array}{ll} (u+v)\cdot w = u\cdot w \ + \ v\cdot w & \forall u,v,w \in V \\ (\alpha u)\cdot v = \alpha(u\cdot v) & \forall \alpha \in F,\ u,v \in V \\ u\cdot v = (\overline{v\cdot u}) & \forall u,v \in V \\ u\cdot u \geq 0 & \forall u \in V \text{ with equality when } u = \vec{0} \end{array}$$

where $\overline{z}$ represents the complex conjugate. In the case of $V = \mathbb{R}^n$, the underlying field is $\mathbb{R}$, and we can simply remove the complex conjugate notation (because every real number is its own complex conjugate) and obtain the following definition of an inner product for a vector space over the reals:

An **inner product** on a vector space V over the field $\mathbb{R}$ is a map

$$\cdot : V \times V \to \mathbb{R}$$

with the following properties:

$$\begin{array}{ll} (u+v)\cdot w = u\cdot w \ + \ v\cdot w & \forall u,v,w \in V \\ (\alpha u)\cdot v = \alpha(u\cdot v) & \forall \alpha \in \mathbb{R},\ u,v \in V \\ u\cdot v = v\cdot u & \forall u,v \in V \\ u\cdot u \geq 0 & \forall u \in V \text{ with equality when } u = \vec{0} \end{array}$$

Note that the only difference from the previous definition is that the inner product is now commutative.

Now in $\mathbb{R}^n$, define the inner product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows.

Definition 1. Suppose $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are arbitrary elements of $\mathbb{R}^n$. Define

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n \ = \ \sum_{i=1}^{n} x_i y_i$$

Theorem 2. $x \cdot y$ is an inner product on $\mathbb{R}^n$

Proof. Let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, and $w = (w_1, \dots, w_n)$ be arbitrary elements (vectors) in $\mathbb{R}^n$, and let $\alpha \in \mathbb{R}$ be an arbitrary scalar.

By definition,

$$(u + v) \cdot w = \sum_{i=1}^{n} (u_i + v_i) w_i$$

The components u_i, v_i, and w_i belong to the scalar field $\mathbb{R}$, so using the algebraic properties of $\mathbb{R}$ we can write

$$(u + v) \cdot w = \sum_{i=1}^{n} (u_i + v_i) w_i \ = \ \sum_{i=1}^{n} u_i w_i + \sum_{i=1}^{n} v_i w_i \ = \ u \cdot w + v \cdot w$$

Also by definition,

$$(\alpha u) \cdot v = \sum_{i=1}^{n} \alpha u_i v_i \ = \ \alpha \sum_{i=1}^{n} u_i v_i \ = \ \alpha (u \cdot v)$$

and

$$u \cdot v = \sum_{i=1}^{n} u_i v_i \ = \ \sum_{i=1}^{n} v_i u_i \ = \ \alpha (v \cdot u)$$

Finally, again making use of the properties of the real numbers,

$$u \cdot u = \sum_{i=1}^{n} u_i^2 \ \geq \ 0 \quad \text{with equality when} \ \ u = \vec{0}$$

□

From a previous set of notes, recall the definition of a norm:

Definition 2 (norm). A nonnegative real-valued function $\| \ \| : V \to \mathbb{R}$ is called a **norm** if:

- $\|v\| \geq 0$ and $\|v\| = 0 \ \Leftrightarrow \ v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F, \ v \in V$

In the current setting, we will take F to be $\mathbb{R}$.

Theorem 3. The positive square root of the inner product of a vector in $\mathbb{R}^n$ with itself is a norm for $\mathbb{R}^n$

Proof. From the previous result,

$$u \cdot u = \sum_{i=1}^{n} u_i^2 \ \geq \ 0 \quad \text{with equality when} \ \ u = \vec{0}$$

so

$$\|u\| = \sqrt{u \cdot u} \geq 0 \quad \text{with equality when} \ \ u = \vec{0}$$

and by definition

$$\alpha u \cdot \alpha u = \sum_{i=1}^{n} \alpha^2 u_i^2$$

so

$$\|\alpha u\| = \sqrt{\alpha u \cdot \alpha u} \ = \ \sqrt{\alpha^2 (u \cdot u)} \ = \ \sqrt{\alpha^2}\sqrt{u \cdot u} = |\alpha| \|u\|$$

Finally, by the definition of the inner product, for any $u, v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$

$$\|u + \alpha v\|^2 \ = \ (u + \alpha v) \cdot (u + \alpha v) \geq 0$$

This can be written as

$$0 \leq u \cdot u + 2\alpha u \cdot v + \alpha^2 \ v \cdot v$$

If $v = \vec{0}$ then the precedint inequality reduces to

$$0 \leq u \cdot u$$

which is true. Otherwise, let

$$\alpha \ = \ -\frac{u \cdot v}{v \cdot v}$$

then

$$0 \leq u \cdot u - 2\frac{u \cdot v}{v \cdot v} u \cdot v + \left(\frac{u \cdot v}{v \cdot v}\right)^2 \ v \cdot v$$

collecting terms we get

$$0 \leq u \cdot u - \frac{(u \cdot v)^2}{v \cdot v} \ = \ \|u\|^2 - \frac{(u \cdot v)^2}{\|v\|^2}$$

which can be rearranged to give

$$(u \cdot v)^2 \ \leq \ \|u\|^2 \|v\|^2$$

which implies that

$$|u \cdot v| \leq \|u\|\|v\|$$

This result is known as the Cauchy-Schwarz inequality, which we will use to establish the third property, the triangle inequality: Suppose $u, v \in \mathbb{R}^n$, then

$$\|u+v\|^2 = (u+v)\cdot(u+v) = u\cdot u + 2\,u\cdot v + v\cdot v$$

$$\leq\ u\cdot u + 2|u\cdot v| + v\cdot v$$

so

$$\|u+v\|^2 \ \leq\ \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \ =\ (\|u\| + \|v\|)^2$$

since all quantities are nonnegative, we can write

$$\|u+v\| \ \leq\ \|u\| + \|v\|$$

which establishes the triangle inequality. □

We have previusly shown that if $\|\cdot\|$ is a norm on a linear space V, then for any $u, v \in V$,

$$\rho(u,v) \ =\ \|u - v\|$$

is a *metric* on V.

In $\mathbb{R}^n$, with the norm $\|v\| = \sqrt{u\cdot u}$, the metric becomes

$$\rho(u,v) = \sqrt{(u-v)\cdot(u-v)} \ =\ \sqrt{\sum_{i=1}^{n}(u_i - v_i)^2}$$

which is known as the *Euclidean* metric or Euclidean distance.

3. Other Norms on $\mathbb{R}^n$

Other norms can be defined in on $\mathbb{R}^n$ in addition to the *Euclidean norm*:

$$\|x\| \ =\ \sqrt{\sum_{i=1}^{n}|x_i|^2}$$

Examples include the l^1 norm,

$$\|x\|_1 \ =\ \sum_{i=1}^{n}|x_i|$$

the sup-norm:

$$\|x\|_\infty \ = \ \max(x_1, \ldots, x_n)$$

and the p-norms for positive integers p:

$$\|x\|_p \ = \ \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

4. Inner Products and Angles

If we picture two vectors u and v in $\mathbb{R}^n$, they are usually thought of as arrows emanating from the origin to two different points.

The vector $u - v$ is pictured as the third side of a triangle, the other two being u and v. The "lengths" of the sides are $\|u\|$, $\|v\|$, and $\|u-v\|$. By the law of cosines, the relationship between the lengths of the sides and angle between the two sides u and v is:

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$$

Alternatively we can write

$$\|u-v\|^2 = (u-v)\cdot(u-v) = u\cdot u + v\cdot v - 2u\cdot v \ = \ \|u\|^2 + \|v\|^2 - 2u\cdot v$$

and equating these two expressions for $\|u - v\|^2$ we get

$$\|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta \ = \ \|u\|^2 + \|v\|^2 - 2u\cdot v$$

or

$$\|u\|\|v\| \cos\theta \ = \ u\cdot v$$

so that

$$\cos\theta \ = \ \frac{u\cdot v}{\|u\|\|v\|}$$

This gives rise to the concept of *orthogonal* vectors as vectors whose inner product is zero.