1. ASSIGNMENT 1

1.1. **Problem 1.** Recall the definition of the *operator norm*: For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, that is, for any $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$,

$$\|T\| = \sup_{\|x\|\neq 0} \frac{\|Tx\|}{\|x\|}$$

Some authors define the operator norm as:

$$\|T\|_{[1]} = \sup_{\|x\|=1} \|Tx\|$$

(Recall that the notation Tx is shorthand for $T(x)$). Show that the two definitions result in the same value.

hint: use some or all of the following facts:

If $A \subseteq B \subseteq [0, \infty)$, then

$$\sup_{x\in A} x \;\leq\; \sup_{x\in B} x$$

$$\left\| \frac{x}{\|x\|} \right\| \;=\; 1 \quad \forall x \in \mathbb{R}^n \setminus 0$$

and, by the properties of linear functions,

$$\frac{1}{\alpha} T(x) \;=\; T\left(\frac{x}{\alpha}\right) \quad \forall \alpha \in \mathbb{R}^+, T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

1.2. **Problem 2.** Show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the matrix

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad 0 \leq \theta < 2\pi$$

preserves lengths, that is to say, for any $x \in \mathbb{R}^2$,

$$\|Bx\| = \|x\|$$

1.3. **Problem 3.** Show that the l^1-norm defined on $\mathbb{R}^n$ by:

$$\|x\|_1 \;=\; \sum_{i=1}^{n} |x_i|$$

satisfies the following conditions for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

- $\|x\|_1 \geq 0$ and $\|x\|_1 = 0 \Leftrightarrow x = \vec{0}$

- $\|\alpha x\|_1 = |\alpha| \|x\|_1$
- $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$

1.4. **Problem 4.** Show that the sup-norm defined on $\mathbb{R}^n$ by:

$$\|x\|_\infty \ = \ \max\{|x_1|, \ldots, |x_n|\}$$

satisfies the following conditions for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

- $\|x\|_\infty \geq 0$ and $\|x\|_\infty = 0 \Leftrightarrow x = \vec{0}$
- $\|\alpha x\|_\infty = |\alpha| \|x\|_\infty$
- $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$