1. ASSIGNMENT 1

1.1. **Problem 1.** Recall the following definitions from the class notes:

Definition (vector space)**.** *A **vector space** or **linear space** consists of:*

- *A field F of elements called **scalars***
- *A commutative group V of elements called **vectors** with respect to a binary operation $+$*
- *A binary operation $: F \times V \to V$ called **scalar multiplication** that associates with each scalar $\alpha \in F$ and vector $v \in V$ a vector αv in such a way that:*
  $1v = v \quad \forall v \in V$
  $(\alpha\beta)v = \alpha(\beta v) \quad \forall \alpha, \beta \in F,\ v \in V$
  $\alpha(v + w) = \alpha v + \alpha w \quad \forall \alpha \in F,\ v, w \in V$
  $(\alpha + \beta)v = \alpha v + \beta v \quad \forall \alpha, \beta \in F,\ v \in V$

Definition (norm)**.** *A nonnegative real-valued function $\| \ \| : V \to \mathbb{R}$ is called a **norm** if:*

- $\|v\| \geq 0$ *and* $\|v\| = 0 \ \Leftrightarrow \ v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality)*
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F,\ v \in V$

Define convergence of a sequence v_n of vectors in a normed linear space V with norm $\| \cdot \|$ as follows:

We say that $\{v_n\}$ *converges* to $v \in V$ and write

$$\lim_{n \to \infty} v_n = v \quad \text{or} \quad v_n \to v \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\|v_n - v\| \ < \ \epsilon \quad \text{for every } n \geq N$$

Suppose u_n and v_n are sequences in a normed linear space V. Show that if

$$\lim_{n \to \infty} u_n = u \quad \text{and} \quad \lim_{n \to \infty} v_n = v$$

and $w_n = u_n + v_n$. Then

$$\lim_{n \to \infty} w_n \ = \ w \ = \ u + v$$

1.2. **Problem 2.** With reference to the definition of convergence in a normed linear space in Problem 1, show that a sequence v_n in a normed linear space V with norm $\|\cdot\|$ can have at most one limit.

1.3. **Problem 3.** Suppose V is a normed linear space with norm $\|\cdot\|$. We say that a sequence of vectors v_n is *bounded* if there exists a real number M such that

$$\|v_n\| \leq M \quad \forall n \in \mathbb{N}$$

With reference to the definition of convergence in a normed linear space in Problem 1, show that if a sequence v_n in a normed linear space is convergent witn $v_n \to v$ as $n \to \infty$, then it is bounded. You may assume $\|v\|$ is finite.

1.4. **Problem 4.** Recall the following definition from the class notes:

Definition (inner product)**.** *Let V be a linear space over the field F, where F is either $\mathbb{R}$ or $\mathbb{C}$. An **inner product** on V is a map*

$$\cdot : V \times V \to \mathbb{F}$$

with the following properties:

$$\begin{array}{ll} (u+v)\cdot w = u\cdot w \ + \ v\cdot w & \forall u,v,w \in V \\ (\alpha u)\cdot v = \alpha(u\cdot v) & \forall \alpha \in F,\ u,v \in V \\ u\cdot v = (\overline{v\cdot u}) & \forall u,v \in V \\ u\cdot u \geq 0 & \forall u \in V \textit{ with equality when } u = \vec{0} \end{array}$$

where $\overline{z}$ denotes the complex conjugate of z, that is, if $z = a + bi$ then $\overline{z} = a - bi$.

Suppose V is an inner product space over $\mathbb{R}$. Show that

$$u \cdot \alpha v \ = \ \alpha(u \cdot v) \quad \forall \alpha \in \mathbb{R},\ u,v \in V$$