BASIC TOPOLOGY OF $\mathbb{R}$ (CONTINUED)

1. Compact Sets

Definition 1 (compact set). A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to an element of K.

Example 1. A closed interval $[a, b]$ is an example of a compact set.

The idea of compactness extends some of the properties that finite sets enjoy to a larger class of possibly infinite sets.

Definition 2 (bounded set). The set $E \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ such that

$$|a| \leq M \quad \text{for all} \quad a \in E$$

Theorem 1 (Heine-Borel Theorem). A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Now we generalize the Nested Interval property to compact sets:

Theorem 2. If $K_1 \supseteq K_2 \supseteq K_3 \cdots$ is a nested sequence of nonempty compact sets, then

$$\bigcap_{n=1}^{\infty} K_n \quad \neq \quad \emptyset$$

Next we consider an important alternative characterization of compact sets in terms of open sets.

Definition 3 (open cover). An **open cover** for a set $E \subseteq \mathbb{R}$ is a collection of open sets

$$\{O_\alpha \ : \ \alpha \in I\}$$

(where I is some index set) with the property that

$$E \subseteq \bigcup_{\alpha \in I} O_\alpha$$

Definition 4 (finite subcover)*.* If

$$\{O_\alpha \ : \ \alpha \in I\}$$

is an open cover of E a **finite subcover** $S = S_1, S_2, \ldots, S_n$ is a finite subcollection of open sets in O that still covers E:

$$E \subseteq S_1 \cup S_2 \cup \cdots \cup S_n$$

Theorem 3*.* For any set $E \subseteq \mathbb{R}$, the following statements are equivalent (that is to say, if any one is true, they are all true):

- E is compact
- E is closed and bounded
- E Every open cover of E has a finite subcover

2. Connected Sets

Definition 5 (separated)*.* Two nonempty sets $A, B \subseteq \mathbb{R}$ are called **separated** if

$$\overline{A} \cap B \ = \ \emptyset \ = \ A \cap \overline{B}$$

Definition 6 (disconnected)*.* A set $E \subseteq \mathbb{R}$ is said to be **disconnected** if it can be written in the form

$$E = A \cup B$$

where A and B are nonempty separated sets. As set which is not disconnected is said to be **connected**.

Definition 7 (totally disconnected)*.* A set E is said to be **totally disconnected** if, for any two distinct elements $a, b \in E$, there exist separated sets A and B with $a \in A$, $b \in B$, and $E = A \cup B$.