BASIC TOPOLOGY OF $\mathbb{R}$

1. Open and Closed Sets

A central idea in topology is the notion of open and closed sets. We will define these and some related terms as they appear in a metric space setting in this section. For now, the only metric space we will consider is $\mathbb{R}$ with the usual metric ρ. Although it suffices as an introduction to the basic ideas in topology, we will not discuss the algebraic properties required for a more thorough introduction to the subject. The subset of topology we are considering is sometimes called *point-set topology*.

Definition 1 (ϵ neighborhood). Given $a \in \mathbb{R}$, an ϵ-**neighborhood** $V_\epsilon(a)$ is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Note that $|x - a|$ is strictly less than ϵ, so an ϵ neighborhood is just an open interval centered at a with radius ϵ.

Definition 2 (open set). The set $E \subseteq \mathbb{R}$ is **open** if every point $a \in E$ has an ϵ neighborhood that is contained in E:

$$V_\epsilon(a) \subseteq E \quad \text{for some} \quad \epsilon > 0$$

Note that $E = \mathbb{R}$ is open, and $E = \emptyset$ is open vacuously.

Also, any interval of the form (a, b) with $a, b \in \mathbb{R}$ and $a < b$ is open.

Theorem 1. Every interval of the form (a, b) with $a, b \in \mathbb{R}$ and $a < b$ is open.

Proof. The interval (a, b) is the same as the set $\{x \in \mathbb{R} : a < x < b\}$. Choose any element $c \in (a, b)$. Then $a < c < b$. Choose ϵ to be the smaller of $c - a$ and $b - c$, that is, take ϵ to be the distance to the closer of a and b. Then the interval $(c - \epsilon, c + \epsilon)$ is contained in (a, b). $\square$

The union of any collection of open sets is also open, as is the intersection of a *finite* number of open sets.

Theorem 2. The union of an arbitrary collection of open sets is open.

Proof. Let I be an index set and suppose

$$S = \bigcup_{\alpha \in I} E_\alpha$$

Given an arbitrary element $s \in S$, we have to show that there is an ϵ neighborhood $V_\epsilon(s)$ that is contained in S. Let $\epsilon > 0$ be given and let $s \in S$ be an arbitrary element of S. Because s is in the union of the E_α, it is in at least one of them, call it E_s. By hypothesis, E_s is open, so by definition there is an ϵ neighborhood $V_\epsilon(s)$ that is entirely contained in E_s. But if $V_\epsilon(s)$ is contained in E_s, by the definition of set union it is also contained in the union of the E_α, which is S. Therefore, S is open. $\square$

Now we consider the intersection of a collection of open sets. As it turns out, we can only guarantee that the intersection will be open if the collection is **finite**.

Theorem 3. The intersection of a finite collection of open sets is open.

Definition 3 (limit point). A point x is a **limit point** of a set E if every ϵ-neighborhood of x intersects E in some point other than x, that is,

$$V_\epsilon \cap E \setminus x \;\neq\; \emptyset \quad \forall \epsilon > 0$$

Example 1. 0 is a limit point of $E = (0, 1)$ because every ϵ-neighborhood $V_\epsilon(0)$ of zero contains the point $\epsilon/2 \in E$, and $\epsilon/2 \neq 0$.

Definition 4 (isoated point). A point x is an **isolated point** of a set E if it is not a limit point of E.

Example 2. Every element z of $\mathbb{Z}$ is an isolated point, because if we choose $\epsilon < 1/2$, there are no elements of $\mathbb{Z}$ in $V_\epsilon(z)$ other than z itself.

Definition 5 (closed set). A set E is said to be **closed** if it contains all of its limit points.

Definition 6 (closure of a set). The **closure** of a set E is the union of E and the set containing all of its limit points.

Definition 7 (perfect set)*.* A set $E \subseteq \mathbb{R}$ is called **perfect** if it is closed and contains no isolated points.

Definition 8 (compliment of a set)*.* The **compliment** of a set E (relative to $\mathbb{R}$) is the set of all real numbers that do not belong to E, that is,

$$E^c = \{x \in \mathbb{R} : x \notin E\} = \mathbb{R} \setminus E$$

Remark 1*.* Note that the properties open and closed are not mutually exclusive. A set can be open, closed, neither open nor closed, or both open and closed. For example, $\mathbb{R}$ is both open and closed. The half-open interval $E = (0, 1]$ is neither open nor closed, because 0 is a limit point of E that does not belong to E, and there are no ϵ-neighborhoods of 1 that are entirely contained in E.

Theorem 4*.* A point x is a limit point of a set E if and only if there exists a sequence $a_n \in E$ such that $a_n \neq x$ for all $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$.

Theorem 5*.* A set E is open if and only if its compliment E^c is closed. A set F is closed if and only if its compliment F^c is open.

Theorem 6*.* The intersection of an arbitrary collection of closed sets is closed. The union of a *finite* collection of closed sets is closed.

Definition 9 (F_σ)*.* A set E is called an F_σ set if it can be represented as a countable union of closed sets.

Definition 10 (G_δ)*.* A set E is called a G_δ set if it can be represented as a countable intersection of open sets.