1. Sequences

1.1. **Note on Remark 1.27.** This remark states that

If $x > 1$ and $x \notin \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $n < x < n + 1$

Proof. By the Archimedean principle (with $a = 1$) there exists an $m \in \mathbb{N}$ such that $x < m$, so the set of all natural numbers greater than x is not empty:

$$E = \{m \in \mathbb{N} : x < m\} \quad \neq \quad \emptyset$$

By the Well-ordering principle, E has a least element $m_0 \in E$, and because m_0 belongs to E, $x < m_0$. This means that

$$m_0 - 1 \leq x$$

since $m_0 - 1$ cannot be in E and be less than m_0, the least element of E.

By hypothesis, $x \notin \mathbb{N}$, and since $m_0 - 1 \in \mathbb{N}$, it must be that $m_0 - 1 \neq x$. Together with the fact that $m_0 - 1 \leq x$, we conclude that

$$m_0 - 1 < x$$

Denoting $m_0 - 1$ by n, we can now write

$$n < x < n + 1$$

□

1.2. **The Squeeze Theorem.**

Theorem (Squeeze Theorem). *Let $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ be real sequences. Suppose $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and there exists an $N_0 \in \mathbb{N}$ such that*

$$x_n \leq w_n \leq y_n \qquad \textit{for } n \geq N_0$$

Then $w_n \to a$ as $n \to \infty$.

Proof. By hypothesis, $x_n \to a$ as $n \to \infty$, so by definition there exists an $N_1 \in \mathbb{N}$ such that

$$|x_n - a| < \epsilon \text{ whenever } n \geq N_1$$

By a similar argument there exists an $N_2 \in \mathbb{N}$ such that

$$|y_n - a| < \epsilon \text{ whenever } n \geq N_2$$

Finally, by hypothesis

$$x_n \le w_n \le y_n \quad \text{whenever } n \ge N_0$$

Choose N to be the largest of N_0, N_1, N_2. Then when $n \ge N$,

$$|x_n - a| < \epsilon \quad \Rightarrow \quad -\epsilon < x_n - a < \epsilon \quad \Rightarrow \quad a - \epsilon < x_n$$

Also

$$|y_n - a| < \epsilon \quad \Rightarrow \quad -\epsilon < y_n - a < \epsilon \quad \Rightarrow \quad y_n < a + \epsilon$$

So, combining inequalities, we have

$$a - \epsilon < x_n \le w_n \le y_n < a + \epsilon$$

which implies that

$$a - \epsilon < w_n < a + \epsilon$$

or

$$-\epsilon < w_n - a < \epsilon \quad \Rightarrow \quad |w_n - a| < \epsilon$$

□

1.3. **Theorem 2.9 ii).**

Theorem (Squeeze Theorem Part ii)). *If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then*

$$x_n y_n \to 0 \quad as \quad n \to \infty$$

Proof. Let $\epsilon > 0$ be given. By hypothesis, $\{y_n\}$ is bounded, so by definition there exists an $M > 0$ such that

$$|y_n| \le M \quad \forall n \in \mathbb{N}$$

Also by hypothesis, $x_n \to 0$ as $n \to \infty$, so there exists an $N \in \mathbb{N}$ such that

$$|x_n - 0| = |x_n| < \frac{\epsilon}{M} \quad \text{whenever } n \ge N$$

Then when $n \ge N$, we have

$$|x_n||y_n| = |x_n y_n| = |x_n y_n - 0| < M \cdot \frac{\epsilon}{M} = \epsilon$$

Since ϵ was an arbitrary choice, we can find such an N for any $\epsilon > 0$, so by definition

$$x_n y_n \to 0 \quad \text{as} \quad n \to \infty$$

□