1. Introduction

Definition (sequence)**.** *A **sequence** is a function* $\{x_n\} : \mathbb{N} \to \mathbb{R}$ *whose domain is* $\mathbb{N}$*. The **range** of the sequence is the set* $\{x_n : n \in \mathbb{N}\}$*.*

We denote a sequence by

$$\{x_n\} \quad \text{or} \quad \{x_n\}_{n=0}^{\infty} \quad \text{or} \quad \{x_n\}_{n\in\mathbb{N}}$$

or possibly by a list of elements

$$x_1, x_2, x_3, \ldots$$

Definition (convergent sequence)**.** *A sequence* $\{x_n\}$ *is said to **converge** to* $a \in \mathbb{R}$ *if, for every* $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that*

$$|x_n - a| < \epsilon \quad \textit{whenever} \quad n \geq N$$

Example. *Prove that* $1/n \to 0$ *as* $n \to \infty$*.*

Proof. Let $\epsilon > 0$ be given. By the Archimedean principle (with $a = 1$), there is an $N \in \mathbb{N}$ such that

$$N > \frac{1}{\epsilon} \quad \text{so that} \quad \frac{1}{N} < \epsilon$$

Then for $n \geq N$,

$$\frac{1}{n} = \frac{1}{n} - 0 = \left|\frac{1}{n} - 0\right| < \epsilon$$

□

Note that if $x_n \to a$ as $n \to \infty$, by definition for any given $\epsilon > 0$, there is an index $N \in \mathbb{N}$ with the property that all terms of the sequence beyond N are within ϵ of a.

Definition (bounded above)**.** *A sequence is said to be **bounded above** if its range is bounded above.*

Definition (bounded below)**.** *A sequence is said to be **bounded above** if its range is bounded below.*

Definition (bounded)**.** *A sequence is said to be **bounded** if it is bounded above and below.*

Theorem. *The limit of a convergent sequence is unique.*

To prove this theorem, we will assume that a sequence converges to two limits, a and b, and show that $|a-b|$ is less than any preassigned $\epsilon > 0$, which by Theorem 1.9 iii), implies that $|a-b| = 0$. This is a type of proof known as an $\epsilon/2$ proof because to make the sum of two quantities less than ϵ, we make each of them less than $\epsilon/2$.

Proof. Let $\epsilon > 0$ be given. By hypothesis $\{x_n\}$ converges to two limits, a and b. Since $x_n \to a$ as $n \to \infty$, by definition there exists an $N_1 \in \mathbb{N}$ such that

$$|x_n - a| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_1$$

Also $x_n \to b$ as $n \to \infty$, so by definition there exists an $N_2 \in \mathbb{N}$ such that

$$|x_n - b| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2$$

Let N be the larger of N_1 and N_2. Then for $n \geq N$,

$$|a-b| = |a - x_n + x_n - b| = |(a - x_n) + (x_n - b)|$$

By the triangle inequality, we can write

$$|(a - x_n) + (x_n - b)| \leq |a - x_n| + |x_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since ϵ was arbitrary, $|a-b|$ can be made smaller than any preassigned $\epsilon > 0$ by taking N sufficiently large, so by Theorem 1.9 iii), $|a-b| = 0$. □

Theorem. *Every convergent subsequence is bounded.*

Proof. By hypothesis, $x_n \to a$ as $n \to \infty$. Choose $\epsilon = 1$. Then by definition there exists an $n \in \mathbb{N}$ such that

$$\text{whenever } n \geq N, \quad |x_n - a| < 1 \Rightarrow -1 < x_n - a < 1 \Rightarrow a-1 < x_n < a+1$$

Now consider x_n for $n < N$. This is a finite set, so let M be the largest absolute value among elements of this set:

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{n-1}|\}$$

Then for $1 \leq n \leq N-1$,

$$-M \leq x_n \leq M$$

Now let m_1 be the smaller of $-M$ and $a-1$, and let m_2 be the larger of M and $a+1$. Then for all $n \in \mathbb{N}$,

$$m_1 \leq x_n \leq m_2$$

so $\{x_n\}$ is bounded. □

Theorem. *Every subsequence of a convergent sequence is convergent, and the limit is the same as the original sequence.*

Proof. Let $\{x_n\}$ be a convergent sequence $x_n \to a$ as $n \to \infty$, and let

$$\{x_{n_k}\}_{k \in \mathbb{N}} \quad \text{with} \quad n_1 < n_2 < n_3 < \cdots$$

Let $\epsilon > 0$ be given, so there exists an $N \in \mathbb{N}$ such that

$$|x_n - a| < \epsilon \quad \text{whenever} \quad n \geq N$$

Note that $n_k > k$ since $n_1 \geq 1$, and if $n_k \geq k$, then $n_{k+1} \geq n_k + 1 \geq k + 1$, so $n_k > k$ for all $k \in \mathbb{N}$, which means that when $n \geq N$, $n_k > n$, and therefore $|x_n - a| < \epsilon$. This establishes that $x_{n_k} \to a$ as $k \to \infty$. □