SECTION 1.1 SUMMARY

1. Basic Definitions

1.1. **Sentences, Variables, and Quantifiers.** The **universal set** is the set containing all objects under consideration. This will usually be determined by the context and may be "all real numbers" or "all natural numbers" or "all US citizens".

A **variable** is a letter used to either:

- 1) Represent a specific element of the universal set
- 2) Hold a place that could potentially be filled by any element of the universal set

A **truth value** is one of the two values true (T) or false (F)

A **sentence** is a declarative sentence that has a truth value.

Sentences fall into one of two categories depending on whether or not they contain a variable:

- A **proposition** is a sentence that does not contain a variable and is simply true or false.
- An **open sentence** is a sentence that contains a variable. The truth value of an open sentence is determined separately for each value of the variable.

Open sentences may be regarded in three ways:

1.1.1. *Case 1: A Collection of Propositions.* The open sentence may be regarded as the collection of propositions obtained by succesively replacing the variable with specific elements of the universal set.

There will be one proposition for each element of the universal set in the collection. Depending on the specific element chosen, some of the propositions in the collection may be true, and some may be false.

In this case the variable will be called a **free variable**

Example 1.1.1. *Suppose the universal set is the set of all Stonehill students. The statement*

$$x \text{ is over 6 feet tall}$$

*is an open sentence that may be regarded as the collection of propositions obtained by replacing x with the name of a current Stonehill student. Some of these propositions will be true, and some will be false, depending on the specific student chosen. In this case, x is calle a **free variable**.*

1.1.2. *Case 2: The Universal Quantifier and Generalizations.* The truth value of the sentence may be regarded as the answer to the question "Is it *always* true?" (i.e., does it become a true proposition when the variable is replaced by an element of the universal set, *regardless of which element is chosen*?).

In this case, the open sentence is said to be **quantified** using the **universal quantifier** "for all" or "for every". The effect of the quantifier is to convert an entire collection of propositions into a single a single proposition that says something about all of them.

An open sentence of this type is called a **generalization**. The quantifier is not always explicitly stated.

Example 1.1.2. *Again, suppose the universal set consists of all Stonehill students. The statement*

$$\text{for all } x \quad x \text{ is over 6 feet tall}$$

which might be rephrased "every Stonehill student is over 6 feet tall" is an open sentence with the universal quantifier. The resulting single proposition is a generalization. In this case, x is a placeholder. If we can produce an exception in the form of single Stonehill student who is not over 6 feet tall, the generalization will be shown to be false. Such an exception is called a counterexample

An **identity** is an equation that, with the addition of the universal quantifier, would become a generalization with a truth value of T, that is, a true generalization.

Example 1.1.3. *The following is an identity:*

$$\cos^2 x + \sin^2 x \;=\; 1$$

The corresponding true generalization would be:

$$\forall x \in \mathbb{R},\ \cos^2 x + \sin^2 x \;=\; 1$$

The symbol $\forall$ *is read "for all" or "for every".*

In the sequel we will consider the the quantifier in such equations to be implied, and will consider an identity to be simply a special kind of true generalization.

1.1.3. *Case 3: The Existential Quantifier and Existence Statements.* The truth value of the sentence may be regarded as the answer to the question "Is it *ever* true?" (i.e., are there *any* elements of the universal set that result in a true propsition when they replace the variable?).

In this case, the open sentence is said to be **quantified** using the **existential quantifier** "there exists" or "for some".

An open sentence of this type is called an **existence statement** Again, the quantifier results in a single proposition that says something about the entire collection.

Example 1.1.4. *The statement*

there exists an x *such that* x *is over 6 feet tall*

which might be rephrased "some Stonehill student is over 6 feet tall" or "at least one Stonehill student is over 6 feet tall" is an open sentence with the existential quantifier. The resulting single proposition is an existence statement. In this case, x *is a placeholder.*

In quantified open sentences (generalizations and existence statements) the variable is called a **placeholder** or **dummy variable** or sometimes a **bound variable**.

1.2. **Axioms, Definitions, and Theorems.** We will consider the structure of Mathematics to be a collection of sentences, beginning with the following:

An **axiom** is a sentence that is accepted as true, for the sake of argument, without being proven.

A **definition** is a sentence that specifies the meaning of a term. Definitions are considered to be inherently true "by definition".

Additional results called **theorems** are added to the collection, but only after they have been shown to be a *logical consequence of prior results*, that is, results **already in the collection**. Axioms and definitions are automatically considered to be part of the collection.

Example 1.2.1. *The Peano axioms for the natural numbers include the following statement:*

1 is a natural number

This is accepted without proof.

Example 1.2.2. *The following statements are definitions:*

If A and B are sets, we say that A is a ***subset*** *of B, denoted by $A \subset B$, if and only if every element of A also belongs to B.*

If A and B are sets, the ***intersection*** *of A and B, denoted by $A \cap B$, is the set consisting of all elements belonging to both A and B.*

Two sets are said to be ***equal*** *if they have exactly the same elements.*

At a given point in time, the collection consists of all axioms, definitions, and theorems for which a valid proof has been given.

For the purpose of proving a new theorem which is not in the collection, the sum total of everything already in the collection is called **prior results**. These are the *only* results that can be assumed in the proof of the new theorem.

Example 1.2.3. *If our collection consisted of the definitions presented in Example 1.2.2, the following statement would be considered a new theorem because it does not appear in the prior results:*

If A and B are sets,

$$A = B \quad \Leftrightarrow \quad A \subset B \text{ and } B \subset A$$

The proof of this theorem in Example 1.2.3 would use the definitions in Example 1.2.2 as prior results.

Until it is proven, a theorem is usually considered to be a **conjecture**, which is a statement that has no connotation of truth or falsity.

Often the first step in such a proof is **translation**, which means replacing terms that have been previously defined with their definitions.

If we used the definitions in Example 1.2.2 to translate the terms in Example 1.2.3, the result would be:

Example 1.2.4. *If A and B are sets,*

$A = B \;\Leftrightarrow\;$ *Every element of A belongs to B and every element of B belongs to A*

1.3. **Logical Form.** The **form** of a (compound) sentence is a symbolic representation where the *connectives* ($\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$) are displayed and the component sentences are replaced by (upper case) letters.

Example 1.3.1. *The logical form of the theorem in example 1.2.3 would be*

$$P \;\Leftrightarrow\; S \wedge T$$

where P is the statement $A = B$, S is the statement $A \subset B$, and T is the statement $B \subset A$.

When the form

$$A \;\Leftrightarrow\; B$$

read "A if and only if B" and abbreviated "A iff B" is true for all values of the variables the sentence contains, the statements A and B are said to be **equivalent**.

Example 1.3.2. *The statements*

$$|x - L| < \epsilon \quad \textit{and} \quad L - \epsilon < x < L + \epsilon$$

are equivalent because

$$\forall x, L \in \mathbb{R} \; \textit{and} \; \epsilon \in \mathbb{R}^+, \quad |x - L| < \epsilon \iff L - \epsilon < x < L + \epsilon$$