OPEN AND CLOSED SETS IN $\mathbb{R}$

1. Basic Definitions and Theorems

A central idea in topology is the notion of an open set. We will define an open set and some related terms as they appear in the context of $\mathbb{R}$ with the usual *metric*, or measure of distance

$$d(x, y) = |x - y|$$

Although it suffices as an introduction to the basic ideas in topology, we will not discuss the algebraic properties required for a more thorough introduction to the subject. The subset we will discuss is sometimes called *point-set* topology.

Definition 1 (topology). A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X that has the following properties:

(1) $\emptyset$ and X are in $\mathcal{T}$
(2) The union of any subcollection of elements of $\mathcal{T}$ is in $\mathcal{T}$
(3) The intersection of any *finite* subcollection of the elements of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**.

We will take the set X to be $\mathbb{R}$. Now we proceed to define an ϵ-neighborhood of a point, which we will use to define an open set. We will show that the open sets that satisfy the definition form a topology on $\mathbb{R}$.

Definition 2 (ϵ neighborhood). Given $a \in \mathbb{R}$, an ϵ-**neighborhood** $V_\epsilon(a)$ is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Note that $|x - a|$ is strictly less than ϵ, so an ϵ neighborhood is just an open interval centered at a with radius ϵ.

Definition 3 (open set). The set $E \subseteq \mathbb{R}$ is **open** if every point $a \in E$ has an ϵ neighborhood that is contained in E:

$$V_\epsilon(a) \subseteq E \quad \text{for some} \quad \epsilon > 0$$

As noted earlier, the collection $\mathcal{T}$ of sets that satisfy this definition of an open set is a topology on $\mathbb{R}$.

Note that $E = \mathbb{R}$ is open, and $E = \emptyset$ is open vacuously.

Also, any interval of the form (a, b) with $a, b \in \mathbb{R}$ and $a < b$ is open.

Theorem 1. Every interval of the form (a, b) with $a, b \in \mathbb{R}$ and $a < b$ is open.

Proof. The interval (a, b) is the same as the set $\{x \in \mathbb{R} : a < x < b\}$. Choose any element $c \in (a, b)$. Then $a < c < b$. Choose ϵ to be the smaller of $c - a$ and $b - c$, that is, take ϵ to be the distance to the closer of a and b. Then the interval $(c - \epsilon, c + \epsilon)$ is contained in (a, b). □

The union of any collection of open sets (finite, countable, or uncountable) is also open, as is the intersection of a *finite* number of open sets.

Theorem 2. The union of an arbitrary collection of open sets is open.

Proof. Let I be an index set and suppose

$$S = \bigcup_{\alpha \in I} E_\alpha$$

Given an arbitrary element $s \in S$, we have to show that there is an ϵ neighborhood $V_\epsilon(s)$ that is contained in S. Let $\epsilon > 0$ be given and let $s \in S$ be an arbitrary element of S. Because s is in the union of the E_α, it is in at least one of them, call it E_s. By hypothesis, E_s is open, so by definition there is an ϵ neighborhood $V_\epsilon(s)$ that is entirely contained in E_s. But if $V_\epsilon(s)$ is contained in E_s, by the definition of set union it is also contained in the union of the E_α, which is S. Therefore, S is open. □

Now we consider the intersection of a collection of open sets. As it turns out, we can only guarantee that the intersection will be open if the collection is **finite**.

Theorem 3. The intersection of a finite collection of open sets is open.

Proof. Let $\{E_1, E_2, \ldots, E_n\}$ be a finite collection of open sets. Let a be an arbitrary element of the intersection:

$$a \in \bigcap_{i=1}^{n} E_i$$

Then by definition $a \in E_i$ for $i = 1, 2, \ldots, n$. Since each E_i is open, for each E_i there exists an ϵ-neighborhood $V_{\epsilon_i}(a)$ of a that is contained in E_i. To show that the intersection is open, we have to find a single ϵ-neighborhood that is contained in **all** of the E_i. Now the ϵ-neighborhoods are open intervals centered at a, so if we take ϵ to be the smallest of the ϵ_i:

$$\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$$

then

$$V_\epsilon(a) \subseteq V_{\epsilon_i} \quad \text{for all } i$$

so

$$V_\epsilon(a) \subseteq \bigcap_{i-1}^{n} E_i$$

Since a was an arbitrary choice, we can find such an ϵ-neighborhood for any a in the intersection, so by definition the intersection is open. □

Together with the assertion that $\mathbb{R}$ and $\emptyset$ are open, the previous two theorems establish that the collection of open sets $\mathcal{T}$ we defined is a topology on $\mathbb{R}$.

There are a great many collections of subsets of $\mathbb{R}$ that qualify as topologies on $\mathbb{R}$, such as the collection consisting only of $\mathbb{R}$ and the empty set. Because of the way this one arises naturally from the metric $d(x, y) = |x - y|$, it is called the *metric topology* on $\mathbb{R}$ with the usual metric.

Definition 4 (limit point). A point x is a **limit point** of a set E if every ϵ-neighborhood of x intersects E in some point other than x, that is,

$$V_\epsilon \cap E \setminus x \neq \emptyset \quad \forall \epsilon > 0$$

Example 1. 0 is a limit point of $E = (0, 1)$ because every ϵ-neighborhood $V_\epsilon(0)$ of zero contains the point $\epsilon/2 \in E$, and $\epsilon/2 \neq 0$.

Definition 5 (isoated point). A point x is an **isolated point** of a set E if it is not a limit point of E.

Example 2. Every element z of $\mathbb{Z}$ is an isolated point, because if we choose $\epsilon < 1/2$, there are no elements of $\mathbb{Z}$ in $V_\epsilon(z)$ other than z itself.

Definition 6 (closed set). A set E is said to be **closed** if it contains all of its limit points.

Definition 7 (closure of a set). The **closure** of a set E is the union of E and the set containing all of its limit points.

Example 3. $C = [0, 1]$ is the closure of $E = (0, 1)$.

Definition 8 (compliment of a set). The **compliment** of a set E (relative to $\mathbb{R}$) is the set of all real numbers that do not belong to E, that is,

$$E^c = \{x \in \mathbb{R} : x \notin E\} = \mathbb{R} \setminus E$$

Remark 1. Note that the properties of being open or closed are not mutually exclusive. A set can be open, closed, neither open nor closed, or both open and closed. For example, $\mathbb{R}$ is both open and closed. The half-open interval $E = (0, 1]$ is neither open nor closed, because 0 is a limit point of E that does not belong to E, and there are no ϵ-neighborhoods of 1 that are entirely contained in E.

Theorem 4. A point x is a limit point of a set E if and only if there exists a sequence $a_n \in E$ such that $a_n \neq x$ for all $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$.

Theorem 5. A set E is open if and only if its compliment E^c is closed. A set F is closed if and only if its compliment F^c is open.

Theorem 6. The intersection of an arbitrary collection of closed sets is closed. The union of a *finite* collection of closed sets is closed.

Definition 9 (F_σ). A set E is called an F_σ set if it can be represented as a countable union of closed sets.

Definition 10 (G_δ). A set E is called a G_δ set if it can be represented as a countable intersection of open sets.