## 1. Limit Superior

**Lemma.** *Suppose $x_n$ is an increasing sequence in $\mathbb{R}$. Then there is an extended real number $x$ such that*

$$x_n \to x \quad \textit{as} \quad n \to \infty$$

*Proof.* By hypothesis, $\{x_n\}$ is increasing. It must be that either $x_n$ is bounded above, or it is not. If it is bounded above, then by the Monotone Convergence Theorem there is an $x \in \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$. Now suppose $x_n$ is not bounded above. Then for any $M \in \setminus$, there is an $N \in \mathbb{N}$ such that $x_N > M$. By hypothesis, $x_n$ is increasing, so we may write

$$x_n \geq x_N > M \quad \text{for all} \quad n \geq N$$

so by definition, $x_n \to \infty$ as $n \to \infty$. □

**Definition** (limit supremum)**.** *Let $\{x_n\}$ be a real sequence. Then the **limit supremum** or **limit superior** of $\{x_n\}$ is the extended real number*

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

*The **limit infimum** or **limit inferior** of $\{x_n\}$ is the extended real number*

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

In Theorem 2.37, the author proves that $\limsup_{n\to\infty} x_n$ is the **largest** subsequential limit of $\{x_n\}$, and $\liminf_{n\to\infty} x_n$ is the **smallest** subsequential limit of $\{x_n\}$.

That is, we can find a subsequence $x_{n_k}$ that converges to $\limsup x_n$ and another subsequence $x_{n_j}$ that converges to $\liminf x_n$, and every other convergent subsequence has the property that

$$\liminf_{n\to\infty} x_n \leq \lim_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n$$

Recall that if a sequence $\{x_{\}}$ converges to a limt $x$, then every subsequence converges to $x$. This means that

**Theorem** (Theorem 2.36)**.**

$$\lim_{n\to\infty} x_n = x \quad \textit{if and only if} \quad \limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$$