LIMITS (CONTINUED)

1. Limit Definitions Summary

1.1. **Two-sided limits.** The two-sided limit definition

$$\lim_{x\to a} f(x) = L$$

requires that the function be defined on either side of a, and intentionally says nothing about the existence or value of $f(a)$. The existence of f on both sides of a is ensured by requiring that a is an element of some open interval I, so that a has to be an interior point of the interval, and the fact that f is required to exist at all points of I except possibly a.

Definition 1 (two-sided limit). Let $a \in I \subseteq I$ where I is an open interval, and let f be a function defined on I except possibly at a. Then we say f converges to L as x approaches a and write

$$\lim_{x\to a} f(x) = L$$

if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

1.2. **Left and right hand limits.** The two one-sided limits make use of an open interval like the two-sided limit, but now the function f is required to be defined at all points of I and a is taken to be one of the endpoints. Since the endpoint is not part of the open interval, the definition says nothing about the existence or value of $f(a)$.

Definition 2 (right-hand limit). Let $I \subset \mathbb{R}$ be an open interval with left endpoint a, and let f be a function defined on I except possibly at a. Then we say f converges to L as x approaches a from the right and write

$$\lim_{x\to a^+} f(x) = L$$

if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad a + \delta \in I \;\text{ and }\; a < x < a + \delta$$

Definition 3 (left-hand limit). Let $I \subset \mathbb{R}$ be an open interval with right endpoint a, and let f be a function defined on I except possibly at a. Then we say f converges to L as x approaches a from the left and write

$$\lim_{x \to a^-} f(x) = L$$

if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad a - \delta \in I \ \text{ and } \ a - \delta < x < a$$

1.3. **Limits at $\pm\infty$.** For the limits as $x \to \pm\infty$, we require that f be defined on an open interval of the form (c, ∞) or $(-\infty, -c)$ that is contained in the domain of f.

Definition 4 (limit as $x \to \infty$). Suppose there is a $c > 0$ such that f is defined on (c, ∞). Then we say f converges to L as x approaches ∞ and write

$$\lim_{x \to \infty} f(x) = L$$

if and only if for every $\epsilon > 0$, there exists an M such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M$$

Definition 5 (limit as $x \to -\infty$). Suppose there is a $c > 0$ such that f is defined on $(-\infty, -c)$. Then we say f converges to L as x approaches $-\infty$ and write

$$\lim_{x \to -\infty} f(x) = L$$

if and only if for every $\epsilon > 0$, there exists an M such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x < M$$

1.4. **convergence to $\pm\infty$.** As with the two-sided limit, we require $a \in I \subseteq \mathbb{R}$ that f be defined on $I \setminus \{a\}$.

Definition 6 (convergence to ∞). Suppose $a \in I \subseteq \mathbb{R}$ and f is defined on $I \setminus \{a\}$. Then we say f converges to ∞ as x approaches a and write

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every $M \in \mathbb{R} > 0$, there exists a $\delta > 0$ such that

$$f(x) > M \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Definition 7 (convergence to $-\infty$). Suppose $a \in I \subseteq \mathbb{R}$ and f is defined on $I \setminus \{a\}$. Then we say f converges to $-\infty$ as x approaches a and write

$$\lim_{x \to a} f(x) = -\infty$$

if and only if for every $M \in \mathbb{R} > 0$, there exists a $\delta > 0$ such that

$$f(x) < M \quad \text{whenever} \quad 0 < |x - a| < \delta$$

1.5. **sequential definition of a limit.** The