1. Two-Sided Limits

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function. We now give a precise definition of the statement

$$\lim_{x \to a} f(x) = a$$

As we will see, the definition has a similar style to the "challenge-response" definition of the limit of a sequence: we suppose we are given an $\epsilon > 0$, and our challenge is to find $N \in \mathbb{N}$ so that

$$|x_n - x| < \epsilon \quad \text{whenever} \quad n \geq N$$

Definition (limit of a function)**.** *Let $I \subseteq \mathbb{R}$ be an open interval and $a \in I$. Suppose f is a real-valued function defined everywhere on I with the possible exception of a. Then we say that $f(x)$ converges to L as x approaches a, and write*

$$\lim_{x \to a} f(x) = L \quad \textit{or} \quad f(x) \to L \quad \textit{as} \quad x \to a$$

if and only if, for every $\epsilon > 0$, there is a $\delta > 0$ (which generally depends on ϵ, f, I, and a) such that

$$0 < |x - a| < \epsilon \quad \textit{implies that} \quad |f(x) - L| < \epsilon$$

A couple of things should be noted about this definition.

First, no assumptions are made about the existence of $f(a)$. It may exist or may not, but in any case it plays no role in the definition of the limit of f as $x \to a$.

Second, the requirement that f be defined on an **open** interval containing a means that $|f(x) - L|$ must be less than some specified ϵ for x within δ of a **on either side** of a. Because an open interval does not include its endpoints, if $a \in I$, we can always choose δ small enough that $(a-\delta, a+\delta) \subset I$. This guarantees that f is defined on $(a-\delta, a+\delta)$, with the possible exception of a.

Example. *Suppose $f(x) = 3x + 5$ for $x \in (-1, 1)$. Use the definition of a function limit to show that $f(x) \to 5$ as $x \to 0$.*

Proof. Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|f(x) - 5| < \epsilon \quad \text{whenever} \quad |x - 0| < \delta$$

First note that

$$|f(x) - 5| = |3x + 5 - 5| = |3x|$$

We need to show that, regardless of how small an ϵ we are given, we can find a δ that guarantees $f(x)$ is within ϵ of L whenever x is within δ of a. The usual way to do this is to find a function

$$f_\delta(\epsilon) = \delta$$

that takes the given ϵ and produces a δ that meets the requirements of the definition of a limit.

This means we want to turn the inequality

$$|f(x) - L| < \epsilon$$

into an inequality of the form

$$|x - a| < \delta$$

In this case, $f(x) = 3x + 5$, and $L = 5$, so we are starting with

$$|3x + 5 - 5| < \epsilon$$

we can simplify this to

$$|3x| < \epsilon \quad \Rightarrow \quad -\epsilon < 3x < \epsilon$$

dividing all terms by 3 gives

$$-\frac{\epsilon}{3} < x < \frac{\epsilon}{3}$$

which we can write as

$$-\frac{\epsilon}{3} < x - 0 < \frac{\epsilon}{3} \quad \text{or} \quad |x - 0| < \frac{\epsilon}{3}$$

So, given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{3}$. Then, working in the other direction, if $|x - 0| < \epsilon/3$,

$$|x - 0| = |x| < \frac{\epsilon}{3} \quad \Rightarrow \quad -\frac{\epsilon}{3} < x < \frac{\epsilon}{3}$$

so, multiplying all terms by 3, we get

$$-\epsilon < 3x < \epsilon \quad \Rightarrow \quad |3x| = |3x + 5 - 5| = |f(x) - 5| < \epsilon$$

Because $\epsilon/3$ is defined for any $\epsilon > 0$, we can always find a $\delta = \epsilon/3$ that guarantees

$$|f(x) - 5| < \epsilon \quad \text{whenever} \quad |x| < \delta = \frac{\epsilon}{3}$$

So, we can say that $\lim_{x\to 0} f(x) = 5$ $\square$

This example and any linear function are easy, because it is easy to convert the inequality involving $|f(x)-L|$ into one involving $|x-a|$. In general, it is not this easy and considerable ingenuity may be required to show that the definition holds.

2. Sequential Characterization of Limits

Theorem (3.6 sequential characterization of limits)**.** *Let $I \subseteq \mathbb{R}$ be an open interval and $a \in I$, and let f be a real-valued function defined on I except possibly at a. Then*

$$\lim_{x \to a} f(x) = L$$

if and only if for every sequence $x_n \in I \setminus \{a\}$ that converges to a, $\{f(x_n)\} \to L$ as $n \to \infty$.