CONTINUITY AND UNIFORM CONTINUITY

1. Continuity

Definition 1 (continuity at a point). Suppose $f : E \to \mathbb{R}$ is a real-valued function defined on E. Then f is said to be **continuous at the point** $a \in E$ if and only if given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad |f(x) - f(a)| < \epsilon$$

Definition 2 (continuity on a set). f is said to be **continuous on** E if and only if f is continuous at every $x \in E$.

Some remarks on this definition are in order.

Note that this definition does **not** say that $\lim_{x\to a} f(x) = f(a)$. The author is apparently trying to handle situations like $f(x) = \sqrt{x}$ which is continuous at $x = 0$ according to this definition, because we only require $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta$ for values of x in $E \subseteq \mathbb{R}$.

This avoids having to define "continuous from the left" and "continuous from the right", using the left and right hand limits. Not all texts use this definition.

The following definition applies when there are no issues with the existence of two-sided limits:

Theorem 1. Let I be an open interval, $a \in I$, and $f : I \to \mathbb{R}$. Then f is continuous at a if and only if

$$\lim_{x\to a} f(x) = f(a)$$

As with function limits, there is a sequential characterization of continuity:

Theorem 2. Suppose $a \in E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. The following statements are equivalent:

i) f is continuous at a

ii) If $\{x_n\}$ converges to a and $x_n \in E \; \forall n \in \mathbb{N}$ then $f(x_n) \to f(a)$ as $n \to \infty$

2. Intermediate Value Theorem

Theorem 3 (intermediate value theorem). Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

3. Uniform Continuity

Definition 3 (uniform continuity). Let $E \subseteq \mathbb{R}$ be a nonempty set and $f : E \to \mathbb{R}$. Then f is said to be **uniformly continuous** if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \quad \text{implies} \quad |f(x) - f(a)| < \epsilon$$

The difference between uniform and ordinary convergence is that in uniform convergence, δ does not depend on x or a. Regardless of which $x, a \in E$ we choose, as long as $|x - a| < \delta$, we are guaranteed that $|f(x) - f(a)| < \epsilon$.

Lemma 1. Suppose $e \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. If $\{x_n\} \in E$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

Proof. Let $\epsilon > 0$ be given. By hypothesis, f is uniformly continuous on E, so there exists a δ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad |x - a| < \delta \quad \forall x, a \in E$$

Also by hypothesis, x_n is Cauchy, so we can treat δ as the given ϵ and so there exists an $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \delta \quad \text{whenever} \quad n, m > N$$

Now let $x_m = a$, so whenever $n, m > N$,

$$|x_n - x_m| < \delta \quad \text{which implies} \quad |f(x_n) - f(x_m)| < \epsilon \quad \text{whenever} \quad n, m > N$$

so $\{f(x_n)\}$ is Cauchy. □

Theorem 4. Suppose I is a closed bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

We will prove this theorem using a proof by contradiction of the *contrapositive*: If f is not uniformly continuous on I, then it is not continuous on I. We will suppose that f is not uniformly continuous on I but f **is** continuous on I, and try to obtain a contradiction. To get a definition of "not uniformly continuous" we negate the definition of uniform continuity to obtain:

$f : E \to \mathbb{R}$ is *not* uniformly continuous on I if there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ we can find two points $x, y \in I$ with the property that

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| > \epsilon_0$$

Since we can find two such points for any $\delta > 0$, we first choose $\delta = 1$. Then we are guaranteed the existence of $x_1, y_1 \in I$ such that $|x_1 - y_1| < 1$ and $|f(x_1) - f(x_2)| > \epsilon_0$. Next, choose $\delta = 1/2$. As before, we are guaranteed the existence of $x_2, y_2 \in I$ such that $|x_2 - y_2| < 1/2$ and $|f(x_2) - f(y_2)| > \epsilon_0$.

$\delta = 1$	$\exists x_1, y_1 \in E$	such that	$\|x_1 - y_1\| < 1$	and	$\|f(x_1) - f(y_1)\| > \epsilon_0$
$\delta = 1/2$	$\exists x_2, y_2 \in E$	such that	$\|x_2 - y_2\| < 1/2$	and	$\|f(x_2) - f(y_2)\| > \epsilon_0$
$\vdots$	$\vdots$				
$\delta = 1/n$	$\exists x_n, y_n \in E$	such that	$\|x_n - y_n\| < 1/n$	and	$\|f(x_n) - f(y_n)\| > \epsilon_0$

Proof. Suppose without loss of generality that $f : [a, b] \to \mathbb{R}$ is continuous, but not uniformly continuous on the closed interval $I = [a, b] \subseteq \mathbb{R}$ with $a < b$. By hypothesis, f is *not* uniformly continuous on I. Then by definition, there exists an ϵ_0 such that for **every** $\delta > 0$ we can find two points $x, y \in I$ such that

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| > \epsilon_0$$

So we can set $\delta = 1$ and find $x_1, y_1 \in E$ such that $|x_1 - y_1| < 1$ and $|f(x_1) - f(y_1)| > \epsilon_0$. Then we can set $\delta = 1/2$ and find $x_2, y_2 \in E$ such that $|x_2 - y_2| < 1/2$ and $|f(x_2) - f(y_2)| > \epsilon_0$.

Continuing in this fashion, we choose $\delta = 1/3, 1/4, 1/5, \ldots$ and produce two sequences x_n and y_n with the property that, for each $n \in \mathbb{N}$,

$$x_n, y_n \in I, \quad |x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| > \epsilon_0$$

Because I is bounded and $x_n, y_n \in I$ for all $n \in \mathbb{N}$, x_n and y_n are bounded and, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $x_{n_i} \to x$. Because

$$a \leq x_n \leq b \quad \forall n \in \mathbb{N}$$

it is also true that

$$a \leq x_{n_i} \leq b \quad \forall i \in \mathbb{N}$$

and therefore by the comparison theorem

$$a \leq x \leq b \quad \forall i \in \mathbb{N}$$

so $x \in I$ and $f(x)$ is defined.

Corresponding to x_{n_i}, there is a subsequence y_{n_i} with the property that $|x_{n_i} - y_{n_i}| < 1/n_i$. Consequently we can choose an $N_1 \in \mathbb{N}$ such that

$$|x_{n_i} - y_{n_i}| < \frac{\epsilon}{2} \quad \text{whenever} \quad i \geq N_1$$

Since $x_{n_i} \to x$, there is an N_2 such that

$$|x_{n_i} - x| < \frac{\epsilon}{2} \quad \text{whenever} \quad i \geq N_2$$

Let N be the larger of N_1 and N_2. Then when $i \geq N$,

$$|y_{n_i} - x| < |y_{n_i} - x_{n_i}| + |x_{n_i} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so $y_{n_i} \to x$ also. By hypothesis f is continuous on I, and we have established that $x \in I$, so f is continuous at x, and by the sequential characteriztion of continuity (3.21 ii),

$$f(x_{n_i}) \to f(x) \quad \text{and} \quad f(y_{n_i} \to f(x)$$

so

$$|f(y_{n_i}) - f(x_{n_i})| \to 0 \quad \text{as} \quad i \to \infty$$

which contradicts the hypothesis that

$$|f(x_n) - f(y_n)| > \epsilon_0 \quad \forall n \in \mathbb{N}$$

□

Theorem 5. Suppose $a < b$ and $f : (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if f can be continuously extended to $[a, b]$, that is, if and only if there is a continuous function g on $[a, b]$ with the property that:

$$f(x) = g(x), \quad x \in (a, b)$$