THE CANTOR SET

1. The Cantor Set

This section describes a set known as the **Cantor Set** because its construction is due to George Cantor. The Cantor set exhibits a number of remarkable properties, some of them quite counterintuitive.

1.1. **Construction.** To construct the Cantor set, we start with the closed interval $C_0 = [0, 1]$ and remove the open interval representing the middle third of the set:

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right)$$

so that

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Now construct C_2 by removing the open intervals representing the middle third of each of the two parts of C_1:

$$C_2 = C1 \setminus \left\{\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right\}$$

So

$$C_2 = \left(0, \frac{1}{9}\right) \cup \left(\frac{2}{9}, \frac{1}{3}\right) \cup \left(\frac{2}{3}, \frac{7}{9}\right) \cup \left(\frac{8}{9}, 1\right)$$

Now continue the process inductively, that is, construct C_3 from C_2, C_4 from C_3, and so on, each time removing the open middle third of each interval in C_n to product C_{n+1}.

The result is a sequence of sets $C_0, C_1, C_2, \ldots$ in which C_n is the union of 2^n closed intervals, each with length $1/3^n$. Now define the Cantor set to be the intersection of this collection:

$$C = \bigcap_{n=0}^{\infty} C_n$$

The Cantor set is the set of numbers that remains after the inductive process of removing the middle thirds of the intervals in the previous step. It is not obvious what is left (if anything), but consider the fact that as we discard an open interval representing the middle third of each closed interval, the endpoints of the original interval are always retained. So 0 and 1 belong to C_0 and carry forward to C_1. They are also endpoints of intervals in C_1, so they carry forward to C_2, and so on, so we can say that at the very least, C contains every one of the endpoints of the closed intervals we produced during its construction. Therefore, not only is the Cantor set not empty, but it is at least countably infinite.

Now consider whether the Cantor set contains anything else. We started with the interval $[0, 1]$ which has length 1, and discarded one interval of length 1/3, two intervals of length 1/9, four intervals of length 1/27, and, in general,

$$2^{n-1} \quad \text{intervals of length} \quad \frac{1}{3^n}$$

The combined lengths of the discarded intervals is

$$L = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1$$

So the total length of the discarded intervals is one, meaning that the Cantor set has zero length. All of this suggests a rather sparse collection consisting of the interval endpoints and not much else.

However, we can present an argument that C is actually uncountable. Suppose we construct an index a_n for each element $x \in C$ as follows: Since C is the intersection of all of the C_i, any point in C belongs to all of them. Starting with C_1, define

$$a_1 = \begin{cases} 0 & \text{if} \quad x \text{ is in the left half of } C_1 \\ 1 & \text{if} \quad x \text{ is in the right half of } C_1 \end{cases}$$

To create C_2, each half of C_1 is split into thirds, and the open middle third discarded. Consequently, if x belongs to, say, the left half of C_1, then when we form C_2 it must belong to either the left or the right half *of the left half* of C_1. So again we have two choices, and we will define, in this case,

$$a_2 = \begin{cases} 0 & \text{if} \quad x \text{ is in the left half of the left half of} C_1 \\ 1 & \text{if} \quad x \text{ is in the right half of the left half} C_1 \end{cases}$$

We continue in this fashion to define $a_3, a_4, a_5 \ldots$ either zero or one depending on whether x lies in the left or right half of its previous interval when that interval is divided. As a result, we end up with a sequence a_n of zeros and ones for each point in C. Because the endpoints are always retained, we know that none of the closed intervals we generate are empty, so there should be at least one point of C belonging to every closed interval in the sequence of nested intervals designated by the sequence of zeros and ones a_n.

So, we expect there are at least as many points in C as there are infinite sequences of zeros and ones. However, the number of sequences of zeros and ones is uncountable. To prove this, consider Cantor's diagonalization argument (Remark 1.39 on page 36 in the text), with the decimal sequences replaced by binary sequences. We construct a sequence not in the list by taking the k^{th} digit of the sequence under construction to be different from the k^{th} digit of the k^{th} sequence in the list.

So we have the remarkable conclusion that although C has length 0, it is uncountable.

In addition, note that since the endpoints of C_0, zero and one, are rational, all of the other endpoints are rational, which means that the set of endpoints is a subset of the rationals Q, and therefore is *at most countable*. So, the vast majority of the points in C belong to C minus the set of endpoints or $C \setminus \{\text{endpoints}\}$, which is uncountable.

Finally, we add that although C contains far more than the endpoints of the intervals, *it does not contain any intervals.* To show this, suppose some interval $I = [a, b] \subseteq C$ where $a, b \in [0, 1]$ and $a < b$. Then every point in I belongs to every one of the C_n, which means $I \subseteq C_n$ for all $n \in \mathbb{N}$. Suppose $\epsilon > 0$ is the length of $I = b - a$. The length of the intervals in C_n is $1/3^n$, so by taking n large enough that

$$n > \log_3 \left(\frac{1}{\epsilon}\right)$$

we have that C_n is shorter than I, which it supposedly contains. The contradiction shows that C does not contain any intervals.