## 1. Recursive Sequences

A number of problems in the text deal with sequences $\left(x_{n}\right)$ defined by a recursion formula and an initial value,

$$
x_{n+1}=f\left(x_{n}\right), \quad n=1,2,3, \ldots \quad \text { with } x_{1}=a
$$

Systems of this type are known as difference equations, discrete dynamical systems, or iterated function systems, among other names.
1.1. Limits of Recursive Sequences. A recursion formula together with an initial value will always produce a unique sequence. One of the questions that arises naturally is that of convergence. Does the resulting sequence converge? If so, how do I find its limit?

One approach would be to use the recursion formula to derive a formula for the $n^{\text {th }}$ term in the sequence, and take the limit of this expression as $n \rightarrow \infty$. Unfortunately, if $f$ is nonlinear, it is usually impossible to find a closed expression for the $n^{\text {th }}$ term, because the expression simply gets more complicated algebraically with each successive term. Most often, some alternative method has to be used.

One approach is to take limits of both sides of the recursion formula as $n \rightarrow \infty$. For example, suppose we are given

$$
x_{n+1}=\frac{1}{2-x_{n}} \quad \text { with } x_{1}=\frac{1}{4}
$$

and asked to find the limit of the resulting sequence (to 4 decimal places),

$$
\frac{1}{4}, \frac{1}{2-\frac{1}{4}}, \ldots=.25, .5714, .7, .7692, .8125, .8421, .8636, \ldots
$$

Intuitively, the sequence appears to be converging, so perhaps the limit exists. Of course, we have to justify this conclusion with a rigorous argument. If we take the limits of both sides of the recursion formula, the result is:

$$
\lim x_{n+1}=\lim \left(\frac{1}{2-x_{n}}\right)=\frac{1}{2-\lim x_{n}}
$$

For the moment, let's set aside the question of whether these limits exist and naively assume that they do, and write

$$
\lim x_{n}=L_{1} \quad \text { and } \quad \lim x_{n+1}=L_{2}
$$

By substitution, we get

$$
\lim x_{n+1}=\frac{1}{2-\lim x_{n}} \quad \Rightarrow \quad L_{2}=1
$$

Rearranging algebraically, we get

$$
-L_{1} L_{2}+2 L_{2}-1=0
$$

Now it seems plausible that $\lim x_{n}$ and $\lim x_{n+1}$ should be the same, because except at the very beginning, the two sequences consist of exactly the same numbers, just labeled differently.

As usual, we have to justify our intuition with a rigorous argument, so consider the following:

Lemma 1. If a sequence is defined recursively by

$$
x_{n+1}=f\left(x_{n}\right), \quad n=1,2,3, \ldots \quad \text { with } x_{1}=a
$$

and
$\lim x_{n}=L$ exists, then $\lim x_{n+1}$ exists and is equal to $L$
(Naturally, the proof is left as an exercise. The generated sequence is:

$$
\left(x_{n}\right)=x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, \ldots
$$

while the sequence $\left(x_{n+1}\right)$ is

$$
\left(x_{n+1}\right)=x_{2}, x_{3}, x_{4}, x_{5}, \ldots
$$

which is a subsequence of the original sequence. Apply theorem 2.5.2).
Now, returning to our equation for the limits of our sample sequence,

$$
-L_{1} L_{2}+2 L_{2}-1=0
$$

in view of the preceding lemma, we can assume that if they exist, $L_{1}$ and $L_{2}$ are the same, so we can write the equation as:

$$
-L_{1}^{2}+2 L_{1}-1=0 \quad \Rightarrow \quad L_{1}=\frac{-2 \pm \sqrt{0}}{-2}=1
$$

so we conclude that

$$
\lim x_{n}=1
$$

which is not at all obvious from the recursion formula, but agrees with the numerical results for the first few terms.
1.2. Existence of Limits of Recursive Sequences. In the preceding section, we proved that if the recursive sequence

$$
x_{n+1}=\frac{1}{2-x_{n}}, \quad n=1,2,3, \ldots, \quad x_{1}=\frac{1}{4}
$$

converges, then

$$
\lim x_{n}=1
$$

That is, we started with the assumption that the limit exists, and then proceeded to show that if it exists, it has to be 1 .

We have not established that the limit exists. Exhibiting numerical values for the first few elements of the sequence might give some idea whether the sequence converges, but it can never prove that it converges, nor can it determine what the limit is.

There is no general method for determining whether a nonlinear recursive sequence of this type converges or not. One method that works for some sequences is the following: if the sequence can be shown to be bounded and monotonic, then the Monotone Convergence Theorem guarantees that it converges.

Claim: The sequence

$$
x_{n+1}=\frac{1}{2-x_{n}}, \quad n=1,2,3, \ldots, \quad x_{1}=\frac{1}{4}
$$

is bounded and monotonic.
Proof. First we will prove that this sequence is monotonic (increasing). The proof will be by induction, so we consider a sequence of propositions of the form:

$$
\begin{array}{lclc}
p(1): & x_{2} & > & x_{1} \\
p(2): & x_{3} & > & x_{2} \\
p(3): & x_{4} & > & x_{3} \\
\vdots & \vdots & & \vdots \\
p(n): & x_{n+1} & > & x_{n} \\
p(n+1): & x_{n+2} & > & x_{n+1}
\end{array}
$$

There are two parts to an induction proof. First, we must prove that $p(1)$ is true. Second, we must show that $p(n) \Rightarrow p(n+1)$, that is, we assume that $p(n)$ is true and try to show that this implies that $p(n+1)$ is true.

We establish $p(1)$ by computation,

$$
x_{2}=\frac{1}{2-x_{1}}=\frac{1}{2-\frac{1}{4}}=\frac{4}{7}>\frac{1}{4}=x_{1}
$$

Now assume that $p(n)$ is true, that is, suppose

$$
x_{n+1}>x_{n}
$$

Then

$$
-x_{n+1}<-x_{n}
$$

and

$$
2-x_{n+1}<2-x_{n}
$$

and finally,

$$
\frac{1}{2-x_{n+1}}<\frac{1}{2-x_{n}} \quad \text { or by substitution, } \quad x_{n+2}<x_{n+1}
$$

Since $p(1)$ is true, and $p(n) \Rightarrow p(n+1)$, by the principle of induction $p(k)$ is true for every $k \in \mathbb{N}$.

Next, we establish that $\left(x_{n}\right)$ is bounded.
We know that $\left(x_{n}\right)$ is increasing, so $x_{1}$ is the smallest element in the sequence, and the sequence is bounded below by $x_{1}=1 / 4$.

All that remains is to show that $\left(x_{n}\right)$ is bounded above. Again, the proof is by induction. This time, the sequence of propositions is:

$$
\begin{array}{lcc}
p(1): & x_{1} & < \\
p(2): & x_{2} & <1 \\
p(3): & x_{3} & <1 \\
\vdots & \vdots & \\
p(n): & x_{n} & <1 \\
p(n+1): & x_{n+1} & <1
\end{array}
$$

Clearly, $p(1)$ is true, because

$$
x_{1}=\frac{1}{4}<1
$$

Now suppose that

$$
x_{n}<1
$$

Then

$$
-x_{n}>-1
$$

and

$$
2-x_{n}>2-1=1
$$

so

$$
\frac{1}{2-x_{n}}<\frac{1}{1}=1
$$

and by substitution

$$
x_{n+1}<1
$$

Since $p(1)$ is true and $p(n) \Rightarrow p(n+1)$, by the principle of induction $p(k)$ is true for every $k \in \mathbb{N}$.

This establishes that $\left(x_{n}\right)$ is bounded and monotonic, so by the Monotone Convergence Theorem, $\left(x_{n}\right)$ converges, and in the preceding section we proved that, if $\left(x_{n}\right)$ converges, then

$$
\lim x_{n}=1
$$

