1. Negation

Definition 1 (negation). A proposition is a **negation** of the proposition A if and only if it is logically equivalent to $\sim A$ ("*not* A").

The truth table for a statement A and its negation $\sim A$ is:

A	$\sim A$
T	F
F	T

The definition requires that the negation be true when the proposition is false, and vice versa.

Equivalently, with this definition it is impossible for a proposition and its negation to be both true, or both false.

If we consider the proposition

A: Smith is the fastest runner on the team,

among the possible ways to negate this statement are:

Smith is not the fastest runner on the team

Some members of the team are faster than Smith

At least one member of the team is faster than Smith

The proposition

Today is Monday

is not the negation of the proposition

Today is Tuesday

because on Wednesday, both are false.

It is always possible to negate a proposition A by prefacing it with the phrase "It is not the case that A", but these trivial negations are seldom useful.

2. Negation of the Conditional

The negation of the conditional statement,

$$\sim (H \Rightarrow C)$$

is logically equivalent to

$$H \wedge (\sim C)$$

Recall that

$$H \Rightarrow C \ \equiv \ (\sim H) \vee C$$

If we negate both sides of this equivalence, we get

$$\sim (H \Rightarrow C) \ \equiv \sim ((\sim H) \vee C)$$

Using the DeMorgan law, we get the result:

$$\sim (H \Rightarrow C) \ \equiv \sim ((\sim H) \vee C) \equiv (\sim (\sim H) \wedge \sim C) \equiv H \wedge (\sim C)$$

Note that the negation of a conditional statement is not a conditional statement.

The following truth table illustrates the fact that $H \wedge (\sim C)$ and $(\sim H) \vee C$ are negations of $H \Rightarrow C$: the truth values in columns for the first two are the exact opposite of those for the third.

H	C	$\sim H$	$\sim C$	$H \Rightarrow C$	$H \wedge (\sim C)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	F

Example 1. Suppose A is the statement

If today is Tuesday, then we have class $\quad [H \Rightarrow C]$

The following statement is a negation of A:

Today is Tuesday and we do not have class $\quad [H \wedge (\sim C)]$

Example 2. Suppose A is the statement

$$\text{If } x \in S \text{ then } x \in T \quad [H \Rightarrow C]$$

The following statement is a negation of A:

$$(x \in S) \wedge (x \notin T) \quad [H \wedge (\sim C)]$$

Example 3. The following statement is often used to define the set of rational numbers $\mathbb{Q}$ using the set of integers $\mathbb{Z}$:

$$\text{If } x \in \mathbb{Q} \ \text{ then } \ \exists p, q \in \mathbb{Z} \ \ni \ x = \frac{p}{q} \qquad\qquad [H \Rightarrow C]$$

This is a conditional or if-then statement with hypothesis or antecedent

$$x \in \mathbb{Q}$$

and conclusion or consequent

$$\exists p, q \in \mathbb{Z} \ \ni \ x = \frac{p}{q}$$

(read "there exist integers p and q such that $x = p/q$").

The following statement is a negation:

$$(x \in \mathbb{Q}) \wedge \sim (\exists p, q \in \mathbb{Z} \ \ni \ x = \frac{p}{q}) \qquad\qquad [H \wedge (\sim C)]$$

This might be read as "x is a rational number but there is no pair integers p, q with $x = \frac{p}{q}$".

Example 4. To prove a conditional statement false, we could show that its negation is true.

Proving the statement

$$H(x) \Rightarrow C(x)$$

to be false is the same as showing

$$H(x) \wedge \sim C(x)$$

is true.

3. Negation of a Generalization

The negation of a generalization is an existence statement.

Example 5. The negation of the statement

$$\text{For every } x \in A, \ x \leq 5$$

is

$$\text{There exists an } \ x \in A \ \ni \ x > 5$$

In general, if x is a variable and $S(x)$ is an open sentence with that variable,

The negation of

$$\text{For all } \ x,\ S(x)$$

is logically equivalent to

$$\text{There exists an } \ x \text{ such that } \ \sim S(x)$$

As a variation we have that the negation of

$$\text{For all } \ x \in T,\ S(x)$$

is logically equivalent to

$$\text{There exists an } \ x \in T \text{ such that } \ \sim S(x)$$

The element $x \in T$ such that $\sim S(x)$ is called a **counterexample** to the generalization. A single counterexample is sufficient to establish that a generalization is false.

To prove a conditional statement false, we could show that its negation is true.

Example 6. Generalizations often contain conditional statements. The following form is particularly common:

$$\forall\, x,\ H(x) \Rightarrow C(x)$$

(read "For all x, if $H(x)$ then $C(x)$" or possibly "For all x, $C(x)$ whenever $H(x)$").

Negating this generalization results in the *existence statement*

$$\exists\, x \ \ni \sim (H(x) \Rightarrow C(x))$$

(read "There exists an x such that $H(x)$ does not imply $C(x)$" or "For some x, $H(x)$ does not imply $C(x)$").

We can use the result on negation of the conditional to write this as:

$$\exists\, x \ \ni \ H(x) \wedge \sim (C(x))$$

(read "There exists an x with $H(x)$ and not $C(x)$" or "For some x, $H(x)$ and not $C(x)$").

4. Negation of an Existence Statement

The negation of an existence statement is a generalization.

Example 7. The negation of the statement

$$\text{There exists an } x \in \mathbb{R} \text{ such that } x^2 < 0$$

is

$$\text{For every } \ x \in \mathbb{R}, \ x^2 \geq 0$$

In general, if x is a variable and $S(x)$ is an open sentence with that variable,

The negation of

$$\text{There exists an } \ x \text{ such that } S(x)$$

is logically equivalent to

$$\text{For all } \ x, \ \sim S(x)$$

As before, we have that the negation of

$$\text{There exists an } \ x \in T, \text{ such that } S(x)$$

is logically equivalent to

$$\text{For all } \ x \in T \ , \sim S(x)$$

The element $x \in T$ such that $S(x)$ is called an **example**. A single example is sufficient to establish that an existence statement is true.

Example 8. The negation of the statement

$$\exists x \in \mathbb{R} \ \ni \ x^2 < 0$$

("There exists a real number x such that $x^2 < 0$")

is

$$\forall x \in \mathbb{R} \ \sim (x^2 < 0)$$

("For every real number x, x^2 is not less than 0")

which could also be written as

$$\forall x \in \mathbb{R} \, x^2 \geq 0$$

5. More Complicated Negations

It is possible to have nested quantifiers. In fact, they are fairly common in statements like the $\epsilon - \delta$ definition of the limit of a function: We say that $\lim_{x \to a} f(x) = L$ if and only if

For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$

To negate this type of sentence, reverse the quantifiers and add *not* to the rightmost sentence:

$$\sim [\forall \epsilon \exists \delta \; S(\epsilon, \delta)] \text{ is equivalent to } \exists \epsilon \forall \delta \sim S(\epsilon, \delta)$$

or

$$\sim [\forall x \in S \exists y \in T \; S(\epsilon, \delta)] \text{ is equivalent to } \exists x \in S \forall y \in T \sim S(x, y)$$

If the existential quantifier appears before the universal quantifier,

$$\sim [\exists \epsilon \forall \delta \; S(\epsilon, \delta)] \text{ is equivalent to } \forall \epsilon \exists \delta \sim S(\epsilon, \delta)$$

or

$$\sim [\exists x \in S \forall y \in T \; S(\epsilon, \delta)] \text{ is equivalent to } \forall x \in S \exists y \in T \sim S(x, y)$$