# THE FUNDAMENTAL THEOREM OF CALCULUS

## 1. The Fundamental Theorem of Calculus

*Theorem* 1 (Fundamental Theorem of Calculus (part 1)). If $f : [a, b] \to \mathbb{R}$ is integrable and

$$F : [a, b] \to \mathbb{R} \quad \text{satisfies} \quad F'(x) = f(x) \text{ on } [a, b]$$

then

$$\int_a^b f(x)dx \quad = \quad F(b) - F(a)$$

*Proof.* Let $P$ be a partition of $[a, b]$. By hypothesis, $F'(x) = f(x)$ on $[a, b]$, so $F$ is differentiable on $[a, b]$ and therefore continuous on $[a, b]$.

Consequently, $f$ is also continuous on each subinterval $[x_{k-1}, x_k]$ of $P$ so by the Mean Value Theorem, there exists a point $t_k \in (x_{k-1}, x_k)$ such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(t_k) = f(t_k)$$

and so

$$F(x_k) - F(x_{k-1}) = f(t_k)(x_k - x_{k-1})$$

Now in each subinterval, let

$$m_k(f) = \inf\{f(x) : x \in (x_{k-1}, x_k)\}$$

and

$$M_k(f) = \sup\{f(x) : x \in (x_{k-1}, x_k)\}$$

then

$$m_k(f) \;\leq\; f(t_k) \;\leq\; M_k(f)$$

and

$$m_k(f)(x_k - x_{k-1}) \;\leq\; f(t_k)(x_k - x_{k-1}) \;\leq\; M_k(f)(x_k - x_{k-1})$$

so

$$\sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) \;\leq\; \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \;\leq\; \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$

but

$$\sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = L(f,P) \quad \text{and} \quad \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = U(f,P)$$

therefore

$$L(f,P) \ \le \ \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \ \le \ U(f,P)$$

By substitution, the middle entry can be written as

$$\begin{aligned} \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \ &= \ \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) \\ &= (F(x_1) - F(a)) + (F(x_2) - F(x_1)) + \cdots \\ &\cdots + (F(x_{n-1}) - F(x_{n-2})) + (F(b) - F(x_{n-1})) \\ &= F(b) - F(a) \end{aligned}$$

and we can write

$$L(f,P) \ \le \ F(b) - F(a) \ \le \ U(f,P)$$

independent of the choice of $P$. Taking limits as $\|P\| \to 0$, by Theorem 5.18

$$\int_a^b f(x)dx \le F(b) - F(a) \le \int_a^b f(x)dx$$

so by the squeeze theorem

$$\int_a^b f(x)dx = F(b) - F(a)$$

□

*Theorem* 2 (Fundamental Theorem of Calculus (part 2))*.* If $f : [a,b] \to \mathbb{R}$ is continuous and

$$G(x) = \int_a^x f(t)dt$$

then $G$ is continuous on $[a,b]$. If $f$ is continuous on $[a,b]$, then $G$ is differentiable on $[a,b]$ and

$$G'(x) = f(x)$$

If $f$ is continuous at $x = c$, then $G$ is differentiable at $x = c$, and $G'(c) = f(c)$.

*Proof.* Suppose

$$G(x) = \int_a^x f(t)dt$$

Then by definition

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

By the properties of integrals,

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt$$

so we can write

$$G(x+h) - G(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$$

and then

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

Because $f$ is continuous on $[x, x+h]$, by the Extreme Value Theorem, there exist $u, v \in [x, x+h]$ at which $f$ attains its minimum $m = f(u)$ and maximum $M = f(v)$ in $[x, x+h]$.

By another property of integrals, if

$$m \leq f(x) \leq M$$

then

$$m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$$

and in this case

$$m(x+h-x) \leq \int_x^{x+h} f(t)dt \leq M(x+h-x)$$

or

$$mh \leq \int_a^b f(t)dt \leq Mh$$

Consider the case where $h > 0$. Then we can divide all entries by $h$ to obtain

$$m \leq \frac{1}{h}\int_a^b f(t)dt \leq M$$

or, equivalently,

$$f(u) \leq \frac{1}{h} \int_a^b f(t)dt \leq f(v)$$

As previously noted,

$$\frac{1}{h} \int_a^b f(t)dt = \frac{G(x+h) - G(x)}{h}$$

so we can substitute for the middle term in the inequality to obtain

$$f(u) \leq \frac{G(x+h) - G(x)}{h} \leq f(v)$$

By construction,

$$x \leq u \leq x + h \quad \text{and} \quad x \leq v \leq v + h$$

So as $h \to 0$, by the squeeze theorem we have

$$\lim_{h \to 0} u = x \quad \text{and} \quad \lim_{h \to 0} v = x$$

Since $f$ is continuous by hypothesis, this implies

$$\lim_{h \to 0} f(u) = f(x) \quad \text{and} \quad \lim_{h \to 0} f(v) = f(x)$$

Taking limits as $h \to 0$ in the inequality gives

$$\lim_{h \to 0} f(u) \leq \lim_{h \to 0} \frac{G(x+h) - G(x)}{h} \leq \lim_{h \to 0} f(v)$$

or

$$f(x) \leq G'(x) \leq f(x)$$

If we assumed $f$ is continuous, this result holds for any $x \in [a, b]$. If we assumed only that $f$ is continuous at a point $x = c$, it holds for $x = c$ (or any other point where $f$ is continuous).

A similar argument holds in the case where $h < 0$. This establishes the result

$$G'(x) = f(x)$$

□