# Assignment 8 Hints

**Problem 2.2.1.** In each of these, use the definition of convergence directly, that is, $\lim(x_n) = x$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \quad \text{whenever} \quad n \geq N$$

Assume you are given some $\epsilon > 0$, and try to solve the inequality

$$|x_n - x| < \epsilon$$

for n. Usually this will result in an expression of the form

$$n > h(\epsilon)$$

for some function $h(\epsilon)$. Then, given any $\epsilon > 0$, take $N > h(\epsilon)$ and the inequality $|x_n - x| < \epsilon$ will hold for all $n > N$.

Example: Show that

$$\lim\left(\frac{n+1}{n+2}\right) = 1$$

In this case the inequality for $\epsilon$ is:

$$\left|\frac{n+1}{n+2} - 1\right| < \epsilon \quad \Rightarrow \quad \left|\frac{n+1}{n+2} - \frac{n+2}{n+2}\right| < \epsilon \quad \Rightarrow \quad \left|\frac{-1}{n+2}\right| < \epsilon$$

$$\Rightarrow \frac{|-1|}{|n+2|} < \epsilon \quad \Rightarrow \quad \frac{1}{n+2} < \epsilon \quad \Rightarrow \quad 1 < n\epsilon + 2\epsilon \quad \Rightarrow \quad \frac{1-2\epsilon}{\epsilon} < n$$

so, given $\epsilon > 0$, taking

$$n > \frac{1-2\epsilon}{\epsilon} \quad \text{guarantees that} \quad \left|\frac{n+1}{n+2} - 1\right| < \epsilon$$

So, take $N$ to be any natural number greater than $(1-2\epsilon)/\epsilon$ (we know there is one by the Archimedean Principle).

**Problem 2.2.4.** Use the definition of convergence directly, that is, $\lim(x_n) = 0$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|x_n - 0| = |x_n| < \epsilon \quad \text{whenever} \quad n \geq N$$

**Problem 2.2.5.** Show that there is an $N \in \mathbb{N}$ such that, if $n > N$, $a_n = 0$.

**Problem 2.2.5b.** This proof is correct.

**Problem 2.2.7.** The idea is to express the fact that $x_n \to \infty$ without specifically referencing $\infty$ in any algebraic expression. In Definition 2.2.3, we take $|x_n - x|$ to be less than any preassigned $\epsilon$. For a sequence that is growing without bound, we want $x_n$ to *exceed* any preassigned $x$, so the definition should say something like:

A sequence $(x_n)$ *converges to infinity* if for every $x \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that

$$x_n > x \quad \text{whenever} \quad n \geq N$$

For specific functions, given a proposed upper bound $x$, try to solve the expression

$$x_n > x$$

for $n$, to get an inequality of the form

$$n > h(x)$$

Then, for any given $x$, take $N > h(x)$.

**Problem 2.2.8a.** This solution is correct.

**Problem 2.3.2a.** We are given $\lim(x_n) = 0$. Given $\epsilon > 0$, the definition of convergence guarantees the existence of an $N \in \mathbb{N}$ that makes $|x_n| < \epsilon^2$. What does this say about $|\sqrt{x_n}|$ when $n > N$?

**Problem 2.3.2b.** We can assume all quantities are positive. Use the "multiplication by conjugates" technique from Calculus I to write

$$|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n} - \sqrt{x}| \left( \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right) = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$$

As in part a), given $\lim x_n = x$ and a value for $\epsilon > 0$, the definition of convergence guarantees the existence of an $N \in \mathbb{N}$ that makes $|x_n - x| < \epsilon \cdot \sqrt{x}$. ($x$ is just a constant). What does this say about $|x_n - x|/\sqrt{x}$ when $n > N$?

**Problem 2.3.3.** Suppose $\epsilon > 0$ is given. We need to find an $N \in \mathbb{N}$ such that $|y_n - l| < \epsilon$ whenever $n \geq N$. Because $x_n, z_n \to l$, there exist $N_1$ and $N_2$ such that, for $n \geq N_1$, $|x_n - l| < \epsilon \Rightarrow x_n \in (l - \epsilon, l + \epsilon)$, and for $n \geq N_2$, $|z_n - l| < \epsilon \Rightarrow z_n \in (l - \epsilon, l + \epsilon)$. Let $N$ be the larger of $N_1$ and $N_2$, and argue that $x_n \leq y_n \leq z_n$ means $y_n \in (l - \epsilon, l + \epsilon) \Rightarrow |y_n - l| < \epsilon$ whenever $n \geq N$.

**Problem 2.3.4.** You can prove this directly from the definition of convergence using what is sometimes called an "$\epsilon/2$" proof. Let $\epsilon > 0$ be given, then by the definition of a limit there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - l_1| < \frac{\epsilon}{2} \text{ wnenever } n \geq N_1 \quad \text{and} \quad |a_n - l_2| < \frac{\epsilon}{2} \text{ wnenever } n \geq N_2$$

Let $N$ be the larger of $N_1$ and $N_2$. Then for $n \geq N$,

$$|l_1 - l_2| = |l_1 - a_n + a_n - l_2| \leq |l_1 - a_n| + |l_2 - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever } n \geq N$$

Now apply Theorem 1.2.6 to conclude that $l_1 = l_2$.

Another possibility is to use the Algebraic Limit Theorem, part (ii), which states (informally) that the limit of the termwise sum of two convergent sequences is the sum of the limits, so let the sequences be $a_n$ and $-a_n$. By Theorem 2.3.3 (i), if $(a_n) \to a$, $\lim -a_n = -\lim a_n = -a$, and

$$\lim(a_n + (-a_n)) = l_1 + (-l_2)$$

but since $a_n - a_n = 0$, this can also be written as

$$\lim(a_n - a_n) = \lim 0 = 0$$

so $l_1 + (-l_2) = 0$ and therefore $l_1 = l_2$.

**Problem 2.3.5.** This is an "if and only if" statement, so we have to prove the implication in both directions.

First, assume $(z_n)$ converges to some limit $l$. Use the definition of convergence and the fact that $y_n = z_{2n}$ to show that $(y_n)$ converges to $l$. Observe that $(x_n = z2n - 1)$. At this point rather than repeat the entire argument with the obvious minor changes, you can simply state that a similar argument shows that $(x_n)$ converges to $l$.

Now assume $(x_n)$ and $(y_n)$ converge to $l$, and let $\epsilon > 0$ be given. Argue that there exist $N_1$ and $N_2$ such that $|x_n - l| < \epsilon$ when $n \geq N_1$ and $|y_n - l| < \epsilon$ when $n \geq N_2$. What can you say about $|z_n - l|$ when $n > \max\{2N_1, 2N_2\}$?

**Problem 2.3.7a.** Use the definition of convergence and the fact the $|a_n| \leq M$ for some $M \in \mathbb{R}$. Suppose we are given $\epsilon > 0$ and note that

$$|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|b_n|$$

We are given that $\lim b_n = 0$, so there exists an $N \in \mathbb{N}$ such that

$$|b_n - 0| = |b_n| < \frac{\epsilon}{M} \quad \text{whenever } n \geq N$$

What does this say about $M|b_n|$?

**Problem 2.3.7b.** Try to construct a counterexample. A good choice for the convergent sequence $(b_n)$ might be $(1, 1, 1, 1, 1, 1 \ldots)$.

**Problem 2.3.7c.** Use the results of part a) and Theorem 2.3.2.

**Problem 2.3.8a.** Divergent doesn't necessarily mean unbounded. Consider a sequence that alternates between two values as a starting point.

**Problem 2.3.8b.** We are given that $(x_n)$ and $(x_n + y_n)$ both converge, so apply the Algebraic Limit Theorem to $x_n \to x$ and $x_n + y_n \to l$:

$$y_n = (x_n + y_n) - x_n \quad \Rightarrow \quad \lim y_n = \lim(x_n + y_n) - \lim x_n$$

What does this say about $\lim y_n$?

**Problem 2.3.8c.** Consider a sequence $(b_n)$ with $\lim(b_n) = 0$.

**Problem 2.3.8d.** Suppose such sequences exist. Apply Theorem 2.3.2 to $(a_n - b_n)$.

**Problem 2.3.8e.** As a candidate for $(a_n)$ consider a sequence that converges to zero.

**Problem 2.3.10.** Use the definition of convergence.

**Problem 2.3.11.** This proof uses the definition of convergence. Suppose $\epsilon > 0$ is given. We have to find $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |y_n - L| < \epsilon \quad \text{for all} \quad n \geq N$$

We know that $\lim x_n = L$, so there exists an $M \in \mathbb{R}$ such that

$$|x_n - L| < M \quad \text{for every } n \in \mathbb{N}$$

We also know that there exists an $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{\epsilon}{2} \quad \text{for every } n \geq N_1$$

Now for every $n \geq N_1$,

$$|y_n - L| = \left| \frac{x_1 + x_2 + \cdots + x_{N_1} + \cdots + x_n}{n} - \frac{nL}{n} \right|$$

$$= \left| \left[ \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} \right] + \left[ \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n} \right] \right|$$

By the triangle inequality,

$$\leq \left| \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} \right| + \left| \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n} \right|$$

$$\leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n}$$

Because $N_1$ and $M$ are fixed but $n$ can be arbitrarily large, there exists $N_2$ such that

$$\frac{(N_1 - 1)M}{n} \leq \frac{\epsilon}{2} \quad \text{for all } n \geq N_2$$

Now let $N = \max\{N_1, N_2\}$. Then

$$\frac{(N_1 - 1)M}{n} \leq \frac{\epsilon}{2} \quad \text{and} \quad \frac{\epsilon(n - N_1)}{2n} \leq \frac{\epsilon}{2} \quad \text{for all } n \geq N$$

and finally

$$|y_n - L| \leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq N$$

For the second part, consider the sequence $(-1^n)$.

**Problem 2.3.12a.** First fix $n$ and let $m \to \infty$, then take the limit of the result as $n \to \infty$, then fix $m$ and let $n \to \infty$, and take the limit of the result as $m \to \infty$.

**Problem 2.3.12b.** The definition is similar to Definition 2.2.3, but the inequality is required to hold when *both* $n$ and $m$ are greater than or equal to $N$.

**Problem 2.4.1.** By hypothesis the sequence of partial sums $(t_k)$ is unbounded, and we can show that the sequence of partial sums $(s_m)$ is unbounded if we can establish that for every $k \in \mathbb{N}$, there is an $m$ for which

$$s_m \geq \frac{t_k}{2}$$

Given an arbitrary $k$,

$$s_{2^k} = b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k})$$

$$\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + \cdots + b_{2^k})$$

$$= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k}$$

$$= \frac{1}{2}\left(2b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}\right)$$

$$= \frac{b_1}{2} + \frac{t_k}{2}$$

so for some $m$,

$$s_m \geq \frac{b_1}{2} + \frac{t_k}{2}$$

The first term on the left is constant, and the second grows without bound. What does this say about $s_m$?

**Problem 2.4.2.** See the recursive sequence example posted on the website.

**Problem 2.4.3.** See the recursive sequence example posted on the website.

**Problem 2.4.4.** Write a recursion formula for the sequence,

$$x_{n+1} = \sqrt{2x_n},\ n = 1, 2, 3, \ldots, \quad x_1 = \sqrt{2}$$

Now we can use induction to determine that $(x_n)$ is increasing. First note that

$$x_1 = \sqrt{2} < \sqrt{2\sqrt{2}} = x_2$$

Now suppose $x_n < x_{n+1}$, then

$$\sqrt{x_n} < \sqrt{x_{n+1}} \quad \Rightarrow \quad \sqrt{2x_n} < \sqrt{2x_{n+1}} \quad \Rightarrow \quad x_{n+1} < x_{n+2}$$

Now use induction to show that $(x_n)$ is bounded above by 2. $x_1 = \sqrt{2} < 2$, and

$$x_n < 2 \quad \Rightarrow \quad 2x_n < 2 \cdot 2 \quad \Rightarrow \quad \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2$$

so $(x_n)$ is bounded above by 2. By the Monotone Convergence Theorem, $(x_n)$ converges, and taking limits of the recursion formula,

$$\lim x_{n+1} = \sqrt{2x_n} \quad \Rightarrow \quad x = \sqrt{2} \cdot \lim \sqrt{x_n}$$

Using Exercise 2.3.2b, if $\lim x_n = x$ then $\lim \sqrt{x_n} = \sqrt{x}$, and finally

$$x = \sqrt{2}\sqrt{x} \quad \Rightarrow \quad x^2 = 2x \quad \Rightarrow \quad x = 2$$

**Problem 2.4.5a.** This problem is similar to the other recursive sequence problems, so we will end up using the Monotone Convergence Theorem to show that the sequence converges, then take limits of both sides of the recursion formula (once we know the limits exist) and solve the resulting equation to find the actual limit. But, in this case the given recursion formula does not lend itself to an easy induction proof for all of the properties we need to establish, so we will have to resort to other methods (there is no general method of showing that a nonlinear recursive sequence converges).

One property we can establish by induction is that, with the given value of $x_1$, $x_n > 0$ for all $n \in \mathbb{N}$: $x_1 > 0$ and, assuming $x_n > 0$,

$$x_n > 0 \quad \Rightarrow \quad x_n + \frac{2}{x_n} > \frac{2}{x_n} \quad \Rightarrow \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) > \frac{1}{x_n} > 0$$

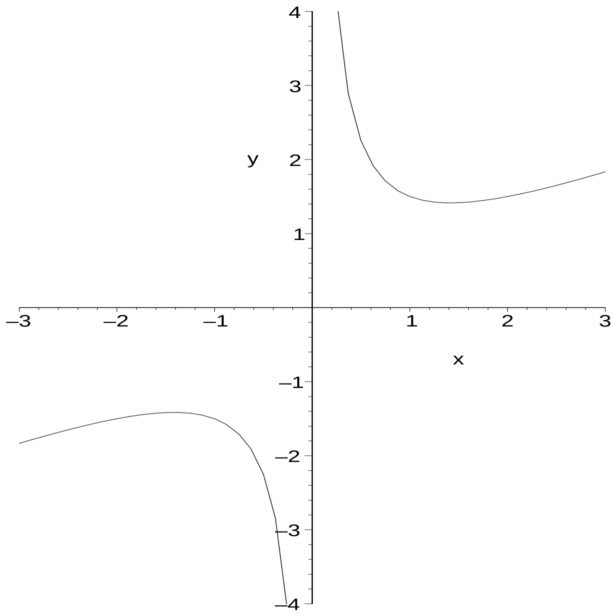
This tells us the generated sequence $(x_n)$ is bounded below by zero. From the first few terms, the sequence appears to be decreasing. If we can show that it is both decreasing and bounded below, the Monotone Convergence Theorem guarantees that it converges.

Unfortunately, it is not obvious how to construct an induction proof that the sequence is decreasing. Here is a direct approach (that is not obvious). From the following graph of

$$f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

it appears that, for positive values of $x$, $\sqrt{2}$ might be a lower bound for $f(x)$:

$$\frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{2} \quad \text{if } x_n > 0$$




Of course, the graph doesn't prove this, so we have to construct a rigorous argument to show that this is the case, but if we knew that $x_n \geq \sqrt{2}$ for all $n$, then we could establish that the sequence $(x_n)$ is decreasing because

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{x_n}{2} - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n} \geq 0$$

Returning to the proposition that $x_n \geq \sqrt{2}$, recall from Calculus I that a function has a minumum at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$. Write the recursion formula as

$$f(x) = \frac{x}{2} + \frac{1}{x} \quad \text{then} \quad f'(x) = \frac{1}{2} - \frac{1}{x^2} \quad \text{and} \quad f''(x) = \frac{2}{x^3}$$

Now observe that this function has a minimum (for positive values of $x$) at $x = \sqrt{2}$, and the minimum function value is

$$\frac{1}{2}\left(\sqrt{2} + \frac{2}{\sqrt{2}}\right) = \sqrt{2}$$

which means that, for positive values of $x_n$,

$$x_{n+1} \geq \sqrt{2} \quad \Rightarrow \quad x_{n+1}^2 \geq 2$$

which is the condition we needed to establish to show that $(x_n)$ is decreasing. Now that we have confirmed that this is the case, the Monotone Convergence Theorem applies and we can be sure that $(x_n)$

converges. It remains only to take limits of both sides of the recursion formula

$$\lim x_{n+1} = \lim \left[\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right] \quad \Rightarrow \quad x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

where $\lim(x_n) = x$. Solving this equation for $x$ yields the final result, $\lim x_n = \sqrt{2}$.

**Problem 2.4.6a.** Define $A_n = \{a_k \ : \ k \geq n\}$, then $y_n = \sup A_n$. Note that $A_{n+1} \subseteq A_n$, and apply the results of Exercise 1.3.4 to show that $y_n$ is decreasing. From the fact that $(a_n)$ is bounded and argue that $(y_n)$ is also bounded. Apply the MCT.

**Problem 2.4.6b.** This time consider $z_n = \inf\{a_k \ : \ k \geq n\}$.

**Problem 2.4.6c.** Argue that for each $n$, in the terminology of parts a) and b)), $y_n \geq z_n$, and apply Theorem 2.3.4.

**Problem 2.4.6d.** This is an if and only if statement, so the implication has to be established in both directions. Suppose $\lim y_n = \lim z_n = l$, and let $\epsilon > 0$ be given. There exists an $N_1 \in \mathbb{N}$ such that $y_n \in V_\epsilon(l)$ for every $n > N_1$, and an $N_2 \in \mathbb{N}$ such that $z_n \in V_\epsilon(l)$ for every $n > N_2$. Let $N = \max\{N_1, N-2\}$. Then for every $n \geq N$, $y_n, z_n \in V_\epsilon(l)$ and since $z_n \leq a_n \leq y_n$ for all $n$, it must be that $a_n \in V_\epsilon(l) \Rightarrow |a_n - l| < \epsilon$ when $n \geq N$.

Next suppose that $\lim a_n = l$. There exists $N \in \mathbb{N}$ such that $a_n \in V_\epsilon(l)$ for every $n \geq N$, so

$$l - \epsilon < a_n < l + \epsilon \quad \text{for every} \quad n \geq N$$

This means that $l - \epsilon$ is a lower bound for the set $\{a_N, a_{N+1}, \ldots\}$ and $l + \epsilon$ is an upper bound for this set. Therefore,

$$l - \epsilon \leq y_n \leq l + \epsilon \quad \text{for every} \quad n \geq N$$

From part a) we know that $\lim y_n$ exists, we can use the Order Limit Theorem and assert that

$$l - \epsilon \leq \lim y_n = y \leq l + \epsilon$$

and, by Theorem 1.2.6, since $\epsilon$ can be made arbitrarily small, $\lim y_n = l$. A similar argument holds for $\lim z_n$.

**Problem 2.5.1.** Suppose $(a_n) \to L$ and $(a_{n_j})$ is a subsequence of $(a_n)$, and let $\epsilon > 0$ be given. We need to find an $N \in \mathbb{N}$ such that

$$|a_{n_j} - L| < \epsilon \quad \text{for all} \quad j \geq N$$

By hypothesis, $(a_n) \to L$, so there exists an $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \quad \text{for all } n \geq N$$

Argue that $n_j \geq j$, that is, $j \geq N \Rightarrow n_j \geq N$, so the same $N$ works for both the original sequence and the subsequence.

**Problem 2.5.2a.** Let $(s_n)$ be the sequence of partial sums,

$$s_n = a_1 + a_2 + \cdots + a_n$$

We are given that $\lim s_n = L$. Now write the regrouped series as:

$$\begin{array}{rcl} b_1 & = & a_1 + a + 2 + \cdots + a_{n_1} \\ b_2 & = & a_{n_1+1} + a + n_1 + 2 + \cdots + a_{n_2} \\ & \vdots & \\ b_m & = & a_{n_{m-1}+1} + a + n_{m-1} + 2 + \cdots + a_{n_m} \\ & \vdots & \end{array}$$

We want to show that the series

$$\sum_{m=1}^{\infty} b_m \quad \text{converges to} \quad L$$

Argue that the sequence of partial sums for the regrouped series,

$$t_m = b_1 + b_2 + \cdots + b_m$$

is just a subsequence of the sequence of partial sums for the original series, and Theorem 2.5.2 applies.

**Problem 2.5.2b.** How does our original series differ from the original series in the examples that were not associative?

**Problem 2.5.3a.** A sequence with the required properties can be constructed by taking any sequence that converges to zero, and any other sequence that converges to one (neither of which is ever 0 or 1), and interleaving the terms. One such example is:

$$x_n = \begin{cases} \frac{1}{n+1} & \text{if} \quad n \text{ is odd} \\ 1 - \frac{1}{n+1} & \text{if} \quad n \text{ is even} \end{cases}$$

**Problem 2.5.3b.** Interleave two sequences with the desired properties.

**Problem 2.5.3c.** Construct a sequence that is frequently in (Exercise 2.2.8) the sets $\{1\}$, $\{1/2\}$, $\{1/3\}$, $\{1/4\}$, etc.

**Problem 2.5.3d.** A sequence with the required properties can be constructed by taking any sequence that converges and interleaving it with an unbounded sequence. One such example is:

$$x_n = \begin{cases} \frac{1}{n} & \text{if} \quad n \text{ is odd} \\ n & \text{if} \quad n \text{ is even} \end{cases}$$

**Problem 2.5.3e.** This result is correct. The bounded subsequence is a sequence in its own right, and by the Bolzano-Weierstrass theorem must have a convergent subsequence, which is in turn a subsequence of the original sequence that converges.

**Problem 2.5.4.** Suppose for the sake of contradiction that $(a_n)$ does not converge to $a$, the limit of every convergent subsequence. Then, negating the definition of a convergent sequence, it must be that there exists an $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$ we can find an $n \geq N$ for which $|a_n - a| \geq \epsilon_0$. Essentially, this says we can construct a subsequence of $(a_n)$ that never enters $V_{\epsilon_0}(a)$. But $(a_n)$ is bounded by hypothesis, so every subsequence of $(a_n)$ is bounded, including the subsequence we constructed, so by the Bolzano-Weierstrass theorem it must have a convergent subsequence, which is also a convergent subsequence of $(a_n)$, and by hypothesis must converge to $a$, but by construction our first subsequence never enters $V_{\epsilon_0}(a)$, a contradiction if it has a subsequence that converges to $a$.

**Problem 2.5.5.** In Example 2.5.3, it was established that for $0 < b < 1$,

$$(b^n) \to 0$$

We wish to extend this result to $-1 < b < 1$. The case $b = 0$ is trivial since

$$(b_n) = 0, 0, 0, \dots$$

so we consider $-1 < b < 0$. Unfortunately $b^n$ is not monotonic, so we cannot use the MCT. By the definition of convergence, $\lim(b_n) = 0$ if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|b_n - 0| < \epsilon \quad \Rightarrow \quad |b^n| < \epsilon \quad \text{whenever} \quad n \geq N$$

For $-1 < b < 0$, we can write $b$ as $-1 \cdot a$ for some $a$ suche that $0 < a < 1$. Then

$$|b^n| = |(-1 \cdot a)^n| = |-1^n||a^n| = |a^n|$$

but, by Example 2.5.3, $|a^n| = a^n \to 0$, so $|b^n| \to 0$ as well, and therefore $\lim b^n = 0$, and we have established that, for $-1 < b < 1$, $\lim b^n = 0$.

**Problem 2.5.6.** Because $(a_n)$ is bounded, the set

$$S = \{x \in \mathbb{R} \ : \ x < a_n \quad \text{for infinitely many terms} \ \ a_n\}$$

is nonempty, and bounded above, so $s = \sup S$ exists.

For any fixed $k \in \mathbb{N}$, there is an $s' \in S$ satisfying

$$s - \frac{1}{k} < s'$$

and, from the way $S$ is defined, $s - 1/k \in S$ and there are an infinite number of $a_n$ satisfying

$$s - \frac{1}{k} < a_n$$

By definition, $s = \sup S$ is an upper bound for $S$, so we can be sure that

$$s + \frac{1}{k} \notin S,$$

(otherwise $s$ would not be an upper bound), which means that there are at most a finite number of elements of $(a_n)$ larger than $s + \frac{1}{k}$. So the fact that an infinite number of terms of $(a_n)$ are greater than $s - \frac{1}{k}$, and only a finite number are greater than $s + \frac{1}{k}$, means that there are an infinite number of terms of $(a_n)$ satisfying

$$s - \frac{1}{k} \leq a_n \leq s + \frac{1}{k}$$

Now we construct a subsequence of $(a_n)$ by choosing an element for each $k \in \mathbb{N}$ as follows. Starting with $k = 1$, choose $a_{n_1}$ so that

$$s - 1 \leq a_{n_1} \leq s + 1$$

Now continue choosing elements $a_{n_k}$ so that each time

$$n_{k+1} > n_k \quad \text{and} \quad s - \frac{1}{k+1} \leq a_{n_{k+1}} \leq s + \frac{1}{k+1}$$

The fact that the inequality is satisfied by an infinite number of terms for any $k$ means we can continue this process indefinitely, and produce a subsequence $(a_{n_k})$ of $(a_n)$.

Now claim that $\lim(a_{n_k}) = s$. To prove this, let $\epsilon > 0$ be given and choose

$$K > \frac{1}{\epsilon}$$

which we can always do by the Archimedean Property. Then if $k \geq K$,

$$\frac{1}{k} < \epsilon \quad \Rightarrow \quad s - \epsilon < a_{n_k} < s + \epsilon \quad \Rightarrow \quad |a_{n_k} - s| < \epsilon$$

for every $k \geq K$.