Problem 1. (4.1.6)

Define $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Find all $n \in \mathbb{N}$ such that $f^{(n)}$ exists for all $x \in \mathbb{R}$.

Problem 2. (4.3.3) $r \in \mathbb{R}$ is called a **root** of a function f if $f(r) = 0$. Show that if f is differentiable on $\mathbb{R}$, then f' has at least one root between any two roots of f.

Problem 3. (4.3.5) Suppose f is differentiable on $\mathbb{R}$. If $f'(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, show that

$$|f(x)| \leq |x| + 1 \quad \text{for all} \quad x \in \mathbb{R}$$

Problem 4. (4.4.2a) Find the Taylor polynomial for $f(x) = \ln x$ centered at $x_0 = 1$

Problem 5. (4.4.1b) Prove that if $x \in [-1, 1]$, then the absolute difference between $\cos x$ and the sum of the first $2n$ terms of the Taylor polynomial for $\cos x$ centered at $x_0 = 0$ is less than or equal to $1/(2n+1)!$:

$$|\cos x - P_{2n}(x)| \leq \frac{1}{(2n+1)!}$$