On the final you will be asked to give one or more of the following proofs:

Theorem 1 (Exercise 3.2.13). The only subsets of $\mathbb{R}$ that are both open and closed are $\mathbb{R}$ and $\emptyset$.

(See class notes for proof)

Theorem 2 (Theorem 3.2.13). A set $O \subseteq \mathbb{R}$ is open if and only if its compliment O^c is closed.

(See class notes for proof)

Theorem 3 (Heine-Borel). A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

(See class notes for proof)

Theorem 4 (Theorem 3.2.3). The union of an arbitrary collection of open sets is open. The intersection of a *finite* collection of open sets is open.

(See class notes for proof)

Theorem 5 (Theorem 3.4.6 $\Leftarrow$). If every pair of nonempty disjoint sets A, B with $A \cup B = E$ has a convergent sequence in A with its limit in B or a convergent sequence in B with its limit in A, then E is connected.

Proof. Unlike open and closed, connected and disconnected *are* mutually exclusive. We will argue the contrapositive which, following the rules of negation, can be stated as:

If E is disconnected, then there exists a pair of nonempty, disjoint sets A, B with $A \cup B = E$ such that no convergent sequence in A has its limit in B, and no convergent sequence in B has its limit in A.

By hypothesis, E is disconnected, so by definition there exist nonempty sets A and B with

$$E = A \cup B \quad \text{and} \quad \overline{A} \cap B = A \cap \overline{B} = \emptyset$$

Let $(a_n) \to a$ be a convergent sequence in A, that is, $a_n \in A$ for every $n \in \mathbb{N}$. If $a_n = a$ for any term of the sequence (a_n), then by hypothesis a has to be in A. If $a_n \neq a$ for every $n \in \mathbb{N}$, then because $(a_n) \to a$, every ϵ-neighborhood of a has to contain an element of (a_n), so it contains an element of A not equal to a, and therefore a is a limit point of A. By definition, if L is the set of limit points of A, the closure of A is

$$\overline{A} = A \cup L$$

so $a \in \overline{A}$. But,

$$\overline{A} \cap B = \emptyset$$

(by hypothesis), so $a \notin B$. A similar argument shows that if $(b_n) \to b$ is a convergent sequence in B, then $b \notin A$, and the result is established. □

Theorem 6 (Exercise 3.2.12e). Every finite set is closed.

Proof. We will show that a finite set F has no limit points, so it satisfies the definition of a closed set vacuously.

Suppose $F \subseteq \mathbb{R}$ is finite, that is, either $F = \emptyset$ or

$$F = \{x_1, x_2, \ldots, x_n\}$$

for some $n \in \mathbb{N}$. If $F = \emptyset$ then F is closed because its compliment, $\mathbb{R}$ is open. If F is a singleton, $F = \{x\}$ for some $x \in \mathbb{R}$, then no ϵ-neighborhood of x (or any other real number) contains elements of F other than x, so F has no limit points.

Finally, suppose F has at least two elements, and let

$$D = \{d_{ij} = |x_i - x_j| \ : \ 1 \leq i \leq n, \quad i \neq j\}$$

That is, $\{d_{ij}\}$ is the set of all distances on the real line between pairs of elements of F. Since F is finite, D is finite as well, so there exists a nonnegative real number d_{min} such that

$$d_{\text{min}} = \min(d_{ij})$$

Now let x be an arbitrary element of F, and choose $0 < \epsilon < d_{\text{min}}$. Then the ϵ-neighborhood

$$V_\epsilon(x) = \{y \in \mathbb{R} \ : \ x - \epsilon < y < x + \epsilon\}$$

does not contain any points of F other than x itself, so x is not a limit point of F. Since x was an arbitrarily chosen, no $x \in F$ is a limit point of F.

□