MA361 Example 1

Theorem 1. There is no rational number whose square is 2.

We will make use of the following *prior results*:

Lemma 1. Every rational number can be written in the form

$$r = \frac{p}{q}$$

where p and q are relatively prime (i.e., p and q have no common divisors).

Proof. The lemma follows immediately from the definition of a rational number. □

Lemma 2. If p is a positive integer and 2 divides p^2, then 2 divides p.

Proof. By the prime factorization theorem, p can be written as a product of prime numbers that is unique up to order:

$$p = a_1^{n_1} a_2^{n_2} \cdots a_j^{n_j}$$

where $a_1, a_2, \ldots, a_j$ are the prime factors of p.

Then p^2 has the same prime factors as p, because

$$p^2 = p \cdot p = \left(a_1^{n_1} a_2^{n_2} \cdots a_j^{n_j}\right)\left(a_1^{n_1} a_2^{n_2} \cdots a_j^{n_j}\right) = a_1^{2n_1} a_2^{2n_2} \cdots a_j^{2n_j}$$

Consequently, if 2 divides p^2, then 2 is a prime factor of p^2. But this implies that 2 is also a prime factor of p, and therefore a divisor of p. □

Proof. (Proof of theorem 1) Suppose for the sake of contradiction that there is a rational number whose square is 2. By the first prior result (lemma 1), this number can be written in the form p/q where p and q are relatively prime (i.e., have no common divisors):

$$2 = \left(\frac{p}{q}\right)^2$$

By the field properties of real numbers (the basic rules of arithmetic),

$$2 = \left(\frac{p}{q}\right) = \frac{p^2}{q^2}$$

so

$$2q^2 = p^2$$

and by the second prior result (lemma 2), 2 is also a divisor of p^2.

Because 2 is a divisor of p, we can write p as

$$2r = p$$

for some integer r. But by the basic rules of arithmetic, this implies that

$$4r^2 = p^2$$

and by substitution using the fact that $p^2 = 2q^2$, we get

$$4r^2 = 2q^2$$

or, by the rules of arithmetic,

$$2r^2 = q^2$$

As before, 2 is a divisor of q by the second prior result (lemma 2). Together with the previous result that 2 is a divisor of p, this contradicts the hypothesis that p and q are relatively prime and the theorem is established. $\square$