MA361 Exam 1

Name: Key (Sample answers)

1) A function $f : A \to B$ is called **onto** if the following statement is true:

For every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

Write a negation of the above statement, that is, a condition under which f is *not* onto.

Statement: $\forall y \in B, \exists x \in A \ni f(x) = y$

Negation: $\exists y \in B \ni \forall x \in A, f(x) \neq y$

2) Let f be a function. Find a statement logically equivalent to the negation of

(f is continuous)$\wedge$(f is monotonic)$\wedge$(f is onto)

that does not use the *AND* connective.

$$\begin{aligned}
\sim(A \wedge B \wedge C) &\equiv \sim((A \wedge B) \wedge C) \\
&\equiv \sim(A \wedge B) \vee \sim C \\
&\equiv (\sim A \vee \sim B) \vee \sim C \\
&\equiv \sim A \vee \sim B \vee \sim C
\end{aligned}$$

"f is not continuous
or f is not monotonic
or f is not ONTO"

3) Find a statement logically equivalent to

If n is a natural number then if $n \neq 2$ then n is not prime.

that uses the *AND* connective.

(Hypothesis in conclusion: $H \Rightarrow (P \Rightarrow C)$

$\equiv (P \wedge H) \Rightarrow C$

"If $n \neq 2$ and n is a natural number then n is not prime"

[note: The fact that the statement is false is irrelevant]

4) We say that two sets A and B have the same **cardinality** if the following statement is true:

There exists a function $f : A \to B$ such that f is $1-1$ and f is onto.

What is the negation of the above statement?

$\exists f: A \to B \ni (f \text{ is } 1\text{-}1) \wedge (f \text{ is onto})$

The negation of an existence statement is a generalization!

$\forall f: A \to B, \sim[(f \text{ is } 1\text{-}1) \wedge (f \text{ is onto})]$ "for every function $f: A \to B$,

$\equiv \forall f: A \to B, \sim(f \text{ is } 1\text{-}1) \vee \sim(f \text{ is onto})$ either f is not 1-1, or f is not onto"

5) Find the contrapositive of the statement:

If $\sqrt{x^2-4} \in \mathbb{R}$ then $x \leq -2$ or $x \geq 2$

$P \Rightarrow Q$

Contrapositive is $\sim Q \Rightarrow \sim P$

$\sim[(x \leq -2) \vee (x \geq 2)] \Rightarrow \sim[\sqrt{x^2-4} \in \mathbb{R}]$ "$-2 < x < 2$ then $\sqrt{x^2-4}$ is not a real number"

$\sim(x \leq -2) \wedge \sim(x \geq 2) \Rightarrow \sqrt{x^2-4} \notin \mathbb{R}$

$(x > -2) \wedge (x < 2) \Rightarrow \sqrt{x^2-4} \notin \mathbb{R}$

6) Write the contrapositive of the statement:

If a set of ordered pairs S defines a function $f : A \to B$ then every element of the set A appears as the first entry of exactly one ordered pair in S.

H: S defines a function $f: A \to B$

C: Every element of A appears as the first entry of exactly one ordered pair in S

Contrapositive

$\sim C \Rightarrow \sim H$

If some element of A appears as the first element of zero or more than one ordered pairs in S, then S does not define a function $f: A \to B$.

7) What is the *converse* of the statement:

If a function f is differentiable at $x = a$, then f is continuous at $x = a$.
H $\Rightarrow$ C

Converse: $C \Rightarrow H$

If f is continuous at $x=a$, Then f is differentiable at $x=a$.

8) Write a statement that does not contain the $\wedge$ connective and is logically equivalent to the statement:

If f is a trigonometric function, then f is continuous on its domain, and if f is an inverse trigonometric function, then f is continuous on its domain.

(Cases) $(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv (P \vee Q) \Rightarrow R$

If f is a trigonometric function or f is an inverse trigonometric function, Then f is continuous on its domain.

9) We say that s is a **least upper bound** for a set A if

(s is an upper bound for A)$\wedge$(for every upper bound b of A, $s \leq b$)

How would you complete the statement "If ...... then s is **not** a least upper bound for A"?

$\sim(P \wedge Q) \equiv \sim P \vee \sim Q$

If (S is not an upper bound for A) or (There exists an upper bound b of A such that $S > b$)

Then S is not a least upper bound for A.

10) Let $a, b \in \mathbb{R}$. Suppose we are given the statement:

If $a = b$, then for every $\epsilon > 0$, $|a - b| < \epsilon$

What is the contrapositive of this statement?

$\sim$[for every $\epsilon > 0$, $|a-b| < \epsilon$] $\Rightarrow$ $\sim$[$a=b$]

If there exists an $\epsilon > 0$ such that $|a-b| \geq \epsilon$, Then $a \neq b$

11) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and $\epsilon, \delta, a, x, L \in \mathbb{R}$. Write a negation of the following statement:

For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, $|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

There exists an $\epsilon > 0$ such That for every $\delta > 0$, there exists an x such that $|x-a| < \delta$ and $|f(x)-L| \geq \epsilon$

12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and $\epsilon, \delta, a, x, L \in \mathbb{R}$. Let A, B be sets. Write the inverse of the statement:

If A is countable and $B \subset A$, then B is either countable or finite.

$(P \wedge Q) \Rightarrow (R \vee S)$

Inverse: $\sim(P \wedge Q) \Rightarrow \sim(R \vee S)$

$\sim P \vee \sim Q \Rightarrow \sim R \wedge \sim S$

If A is not countable, or B is not a subset of A, Then B is not countable and B is not finite.

13) Let A, B be sets. Supply a hypothesis for the conditional statement

$$\ldots\ldots \Rightarrow x \notin A \cap B$$

that makes it logically equivalent to the statement

$$x \in A \cap B \Rightarrow (x \in A) \wedge (x \in B)$$

$$P \Rightarrow Q$$

$$\sim Q \Rightarrow \sim P$$

$$\sim[(x \in A) \wedge (x \in B)] \Rightarrow \sim[x \in A \cap B]$$

$$\sim(x \in A) \vee \sim(x \in B) \Rightarrow x \notin A \cap B$$

If x is not in A or x is not in B, Then x is not in $A \cap B$.

14) Suppose $S = \{x_n\}_{n=1}^{\infty}$ is a sequence. We say that S is **bounded** if the following statement is true:

There exists an $M \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $|x_n| \leq M$.

We say that S is **unbounded** if the negation of the above statement is true. What is the negation of this statement?

S: $\exists M \in \mathbb{R} \ni \forall n \in \mathbb{N}, |x_n| \leq M$

Negation: $\forall M \in \mathbb{R}, \exists n \in \mathbb{N} \ni |x_n| > M$

For every real number M, There exists an $n \in \mathbb{N}$ such that $x_n > M$.

15) Let $\mathbb{Z}$ be the set of integers and $\mathbb{Q}$ be the set of rational numbers. We say that a real number x belongs to $\mathbb{Q}$ if the following statement is true:

There exist $n, m \in \mathbb{Z}$ such that $x = n/m$.

We say that x is **irrational** if the negation of the above statement is true. What is the negation of this statement?

S: $\exists n, m \in \mathbb{Z} \ni x = n/m$

Negation: for every $n, m \in \mathbb{Z}$, $x \neq n/m$